SCIENCE IS WHAT WE UNDERSTAND WELL ENOUGH TO EXPLAIN TO A COMPUTER. ART IS EVERYTHING ELSE WE DO. DURING THE PAST SEVERAL YEARS AN IMPORTANT PART OF MATHEMATICS HAS BEEN TRANSFORMED FROM AN ART TO A SCIENCE: NO LONGER DO WE NEED TO GET A BRILLIANT INSIGHT IN ORDER TO EVALUATE SUMS OF BINOMIAL COEFFICIENTS, AND MANY SIMILAR FORMULAS THAT ARISE FREQUENTLY IN PRACTICE; WE CAN NOW FOLLOW A MECHANICAL PROCEDURE AND DISCOVER THE ANSWERS QUITE SYSTEMATICALLY

DONALD E. KNUTH

DEAR JAAP, GOOD LUCK WITH YOUR PROBLEM-SOLVING! PROBLEM-SOLVERS ARE AN ENDANGERED SPECIES, AND IT IS NICE THAT PEOPLE LIKE YOU KEEP THE FLAME ALIVE. BEST WISHES

DORON ZEILBERGER

MY MOTTO FOR LIFE: "DER JAAP IST BRILLANT, ABER FAUL", WHICH TRANSLATES INTO: "JAAP IS BRILLIANT, BUT LAZY".

HANS FREUNDENTHAL

TO MANY LAYMEN, MATHEMATICIANS APPEAR TO BE PROBLEM SOLVERS, PEOPLE WHO DO "HARD SUMS". EVEN INSIDE THE PROFESSION WE CLASSIFY OURSELVES AS EITHER THEORISTS OR PROBLEM SOLVERS. MATHEMATICS IS KEPT ALIVE, MUCH MORE THAN BY THE ACTIVITIES OF EITHER CLASS, BY THE APPEARANCE OF A SUCCESSION OF UNSOLVED PROBLEMS, BOTH FROM WITHIN MATHEMATICS ITSELF AND FROM THE INCREASING NUMBER OF DISCIPLINES WHERE IT IS APPLIED. MATHEMATICS OFTEN OWES MORE TO THOSE WHO ASK QUESTIONS THAN TO THOSE WHO ANSWER THEM.

RICHARD K. GUY

JAAP SPIES

A BIT OF MATH

THE JOY OF PROBLEM SOLVING

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Contents

Dedicated to all friends of Mathematics

and Good Thinking

Introduction

Solving Math Problems

You better read this introduction on my website www.jaapspies.nl, unless you are reading a pdf with link information.

In the first epigraph I cite Donald E. Knuth: "Science is what we understand well enough to explain to a computer. Art is everything else we do." With this statement in mind I made up the subtitle of this book: The Art of Problem Solving. Art is a simple word with many meanings. I don't feel very comfortable with the idea of doing 'art', or being an artist. Problem Solving is work, sometimes hard work, mostly a pleasant job. And in the end there is a reward, sometimes.

See also Knuth: The Art of Computer Programming

The joy of a discovery, to know something you didn' knew before. And above all it is learning in a pleasant way. I remember the feeling when I could write QED at the end of a proof of a theorem in Elementary Geometry. In the early sixties I enjoyed the 'Practica' alongside the lectures on Linear Algebra of Hans Freudenthal and on Analysis of Fred van der Blij. Making exercises, doing homework and getting feedback from student assistants. In my third year at Utrecht University I became an assistant myself.

You may have noticed. The subtitle on the cover differs from that on the inside. The flag doesn't cover the load!

One of the textbooks I like most is Graham, Knuth and Patashnik: Concrete Mathematics.[1] The book contains more than 500 exercises from 'warmups', 'basics', 'exam problems' to 'research problems'. Answers and extra information is found in an Appendix. The answers to the research problems are not complete! The marginnotes are amusing.

[1] Ronald Graham, Donald Knuth, and Oren Patashnik. *Concrete mathematics : a foundation for computer science*. Addison-Wesley, Reading, Mass, 1994. ISBN 0201142368

Problem solving, doing exercises, is an essential part of Mathematical Education. The importance of a 'Problem Section' in Mathematical Journals should not be underestimated. Read 'Opinion 19' from Doron Zeilberger as he warns the SIAM Review in 1997 for discontinuing its Problem Section. He noted: "Problem sections turned many young people into mathemati-

Figure 1: Doron Zeilberger: Opinion 19

cians". His first 'publication' was a solved problem in SIAM Review. "Both the problem and my solution were real gems", he said, "I'll forget my right hand before I'll forget it".

Not only young people profit from the existence of Problem Sections. Near the end of my professional career I found my way back to Mathematics. Starting in December 2001 with a simple formulated problem: 'Does there exist a triangle with sides of integral length such that its area is equal to the square of the length of one of its sides'.

I found this problem as Problem 26 in the Problem Section of the 'Nieuw Archief voor Wiskunde' (NAW).

Figure 2: NAW December 2001

NAW

The NAW is a publication of the Dutch Royal Mathematical Society, the Koninklijk Wiskundig Genootschap (KWG). The society publishes the Nieuw Archief voor Wiskunde (NAW), a beautiful quarterly for all of its members with a famous Problem Section. The Problem Section has a very old history. The Problems and Solutions have been published since the early beginnings of the Society in 1778, and for several decades they constituted the most substantial part of the material in the Proceedings of the Society. The KWG is the world's oldest national Mathematical Society.

https://www.nieuwarchief.nl

My Solution of Problem 26

In the Newsgroup sci.math I discovered an article by Dave Rusin, linking the existence of certain triangles with rational sides to elliptic curves. From there I found The Mathematical Atlas 14H52: Elliptic Curves. A real treasure trove! Learning from the well written lecture notes of J.S Milne on Elliptic Curves (1996), finding the Maple package APECS from Ian Connell, the ec-tables of John Cremona, I came up with a solution. See chapter 30, Solution 1. There are no such triangles.

Figure 3: Mathematical Atlas 14H52 (Wayback Machine)

This was the start of finding a series of solutions. The latest was Solution 8 in July 2005, based on a parametric representation of Heronian triangles found in my solution of Problem 2005-1/C

(Chapter 14).

You can find the published solution of Frits Beukers in the NAW 5/3 nr. 3 and my collection of solutions can be found on my website. And of course here in this book as Chapter 30.

Figure 4: NAW September 2002
See my website: https://www.jaapspies.nl/mathfiles

Next Problem

The next problem that touched me by surprise was Problem 29 of the NAW. There was an erroneous solution in NAW 5/3 nr. 3 and the problem was declared open again in NAW 5/3 nr. 4. The editor of the Problem Section Robbert Fokkink challenged me to attack this unsolved problem. I found a solution in the beginning of Januari 2003. Interesting is to know that Problem 29 originated from work related to a paper of Lute Kamstra: Juggling polynomials, CWI Report PNA-R0113, July, 2001.

Figure 5: NAW December 2002

For myself I translated and extended the problem to a Dancing School Problem: "How to match boys and girls in a dancing class under certain length restrictions". My story of the Dancing Schools includes the solution of Problem 29, but also links with certain kinds of Rook Placing Problems. There is a SAGE-program to generate polynomial solution to a certain class of problems. See also Chapter 32. From the Dancing School Problem originated the sequences A079908-A079928 from the OEIS.

https://oeis.org

My solution of problem 29 is in terms of the permanent of (0,1)-matrices. So I became interested in algorithms for permanents. Playing with small examples and counting I found an alternative for the famous Ryser's algorithm. See Chapter 33. I implemented my algorithm in a C/C++program, which was used to contribute to Neil Sloane's On-line Encyclopedia of Integer Sequences (OEIS). See for instance A087982, A088672 and A089476.

Problem 29 showed up in disguise as part 2 of Problem2006-2B (see below) in the NAW 5/7 nr. 2. There were no solutions sent in, so this is an absolute waste of a nice problem!

Figure 6: Dancing School Problems NAW Dec 2006

There is an article in the NAW 5/7 nr. 4 December, 2006: Dancing School problems, Permanent solutions of Problem 29. In a sidenote of this paper was the first appearance of my formula for the permanent of a square matrix. In the Dutch Wikipedia on

Permanents known as the ’Formule van Spies’.

UWC/Problems

The Problem Section of the NAW was discontinued and merged with the UWC, the University Math Competition, open for Belgian and Dutch math students. Starting with NAW 5/4 nr.1 the Problem Section and the UWC became the section Problemen/UWC. First in Dutch, but later on my suggestion the problems are formulated in the English language, as are the solutions. Students can gather points with their solutions. Others can send their solutions ‘hors concours’.

In the NAW 5/5 nr. 3 there was no UWC/Problems section, due to a misunderstanding between the editorial board and the editors of the section. There was a change of editors starting with the NAW 5/5 nr. 4. Note the difference: Opgave is replaced by Problem.

The UWC has changed back into a general Problem Section, open to everyone.

My latest submission was a partial solution to Problem 2020-1/C. You will not find it here in this book.

Figure 7: Archive of Problems and Solutions since 2005

Personal Remarks

I would like to thank all the editors of the Problem Section. Especially Robbert Fokkink, who is now the Editor in Chief of the Nieuw Archief van de Wiskunde. Without him my way back to Mathematics would have been non-existent.

I wrote this book for myself. Just to see on paper what Problem Solving has brought me. I really enjoyed the work and the insights I got doing it.

Part I

Problems from the NAW

1

Opgave A NAW 5/4 nr. 3, September 2003

The problem

Introduction

Let $\{a_n\}_{n=0}^{\infty}$ be a non-decreasing sequence of real numbers such that
$(n-1)a_n = na_{n-2}$ for $n = 1, 2, ...$ with initial value $a_0 = 2$. We have to calculate a_1.

Solution 1

We have $a_{2k-2} \leq a_{2k-1} \leq a_{2k}$ for $k \geq 2$.
The recursion $(n-1)a_n = na_{n-2}$ leads to the following results:

For $n = 2k-1$

$$a_{2k-1} = \frac{2k-1}{2k-2} \cdot \frac{2k-3}{2k-4} \cdot ... \cdot \frac{3}{2} \cdot a_1$$

and for $n = 2k$

$$a_{2k} = \frac{2k}{2k-1} \cdot \frac{2k-2}{2k-3} \cdot ... \cdot \frac{2}{1} \cdot a_0 = 2k \cdot \frac{2k-2}{2k-1} \cdot \frac{2k-4}{2k-3} \cdot ... \cdot \frac{2}{3} \cdot \frac{1}{1} \cdot a_0$$

This can be written with double factorials

as

$$a_{2k-1} = \frac{(2k-1)!!}{(2k-2)!!} \cdot a_1$$

and

$$a_{2k} = 2k \cdot \frac{(2k-2)!!}{(2k-1)!!} \cdot a_0$$

Figure 1.1: QR-code wikipedia

So

$$(2k-1)\cdot\frac{((2k-2)!!)^2}{((2k-1)!!)^2}\cdot a_0 \le a_1 \le 2k\cdot\frac{((2k-2)!!)^2}{((2k-1)!!)^2}\cdot a_0$$

As we can easily see

$$a_1 = \lim_{k\to\infty} 2k\cdot\frac{((2k-2)!!)^2}{((2k-1)!!)^2}\cdot a_0$$

From the properties of double factorials it follows that

$$(2k-2)!! = 2^{k-1}\cdot(k-1)! = 2^{k-1}\cdot\Gamma(k)$$

and

$$(2k-1)!! = \frac{2^k}{\sqrt{\pi}}\Gamma(\frac{1}{2}+k)$$

So with a well known limit we get

$$a_1 = \lim_{k\to\infty}\pi\cdot\frac{k(\Gamma(k))^2}{(\Gamma(\frac{1}{2}+k))^2} = \pi\cdot 1 = \pi$$

Solution 2

Without double factorials we can write

$$a_{2k-1} = \frac{(2k-1)!}{(2^{k-1}\cdot(k-1)!)^2}\cdot a_1$$

and

$$a_{2k} = \frac{2k(2^{k-1}(k-1)!)^2}{(2k-1)!}\cdot a_0$$

and hence

$$a_1 = \lim_{k\to\infty}\frac{2k(2^{k-1}(k-1)!)^4}{((2k-1)!)^2}\cdot a_0 = \lim_{k\to\infty}\frac{2k\cdot 2^{4(k-1)}}{(2k-1)^2\binom{2k-2}{k-1}^2}\cdot a_0$$

Writing $a_0 = 2$ and $n = k-1$ we get with other well known limits

$$a_1 = \lim_{n\to\infty}\frac{4(n+1)\cdot 2^{4n}}{(2n+1)^2\binom{2n}{n}^2} = \lim_{n\to\infty}\frac{4n^2+4n}{4n^2+4n+1}\cdot\frac{2^{4n}}{n\binom{2n}{n}^2} = 1\cdot\pi = \pi$$

2

Opgave B NAW 5/4 nr. 3, September 2003

The problem

Introduction.

Let $S(n)$ be the sum of the remainders on division of the natural number n by $2, 3, ..., n-1$. Show that

$$\lim_{n\to\infty} \frac{S(n)}{n^2}$$

exists and compute its value.

Solution.

We define the remainder in the division of n by k by $r_{n,k} = n - k \cdot [\frac{n}{k}]$, so from the definition of $S(n)$ it follows that

$$S(n) = \sum_{k=2}^{n-1} r_{n,k} = \sum_{k=1}^{n} r_{n,k} = \sum_{k=1}^{n} (n - k \cdot [\frac{n}{k}]) = n^2 - \sum_{k=1}^{n} k \cdot [\frac{n}{k}]$$

With $\sigma(k) = \sum_{d|k} d$, Theorem 324 and the proof of this Theorem taken from Hardy and Wright, An Introduction to the Theory of Numbers, 5th ed. p. 264-266[1], we get

$$\begin{aligned} S(n) &= n^2 - \sum_{x=1}^{n} \sum_{1 \le y \le n/x} y = n^2 - \sum_{k=1}^{n} \sigma(k) \\ &= n^2 - (\frac{1}{12}\pi^2 n^2 + O(nlogn)) \end{aligned}$$

Hence

$$\lim_{n\to\infty} \frac{S(n)}{n^2} = 1 - \frac{1}{12}\pi^2$$

[1] G.H. Hardy and E.M. Wright. *An introduction to the theory of numbers*. Clarendon Press Oxford University Press, Oxford New York, 1979. ISBN 0198531710

3
Opgave C NAW 5/4 nr. 3, September 2003

The problem

Introduction.

See NAW 5/4/ nr. 3 September 2003, opgave C:

https://www.nieuwarchief.nl/serie5/pdf/naw5-2003-04-3-269.pdf

Figure 3.1: QR-code link

Solution.

Let the points A, B, C, D be defined by coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4). As easily can be shown, we have $Z_1 = Z_2$ with $x_{Z_1} = x_{Z_2} = \frac{1}{4}\sum_{i=1}^{4} x_i$ and $y_{Z_1} = y_{Z_2} = \frac{1}{4}\sum_{i=1}^{4} y_i$.

Let S_1 be the centre of gravity of ABD and S_2 that of BCD, than we have
$x_{S_1} = \frac{x_1+x_2+x_4}{3}$, $y_{S_1} = \frac{y_1+y_2+y_4}{3}$, $x_{S_2} = \frac{x_2+x_3+x_4}{3}$ and $y_{S_2} = \frac{y_2+y_3+y_4}{3}$.

We define A_1 as the 'area' of ABD, A_2 the 'area' of BCD and the weighting factors $p_1 = \frac{A_1}{A_1+A_2}$ and $p_2 = \frac{A_2}{A_1+A_2}$.

According to a more or less well known result we can calculate A_1 and A_2 from the coordinates:
$A_1 = \frac{1}{2}((y2 - y1)(x_1 + x_2) + (y_4 - y_2)(x_4 + x_2) + (y_1 - y_4)(x_1 + x_4))$ and
$A_2 = \frac{1}{2}((y3 - y2)(x_3 + x_2) + (y_4 - y_3)(x_4 + x_3) + (y_2 - y_4)(x_2 + x_4))$.

See Lemma below

Now we can calculate Z_3 with $x_{Z_3} = p_1 \cdot x_{S_1} + p_2 \cdot x_{S_2}$ and $y_{Z_3} = p_1 \cdot y_{S_1} + p_2 \cdot y_{S_2}$.

If $Z_1 = Z_2 = Z_3$ we have equations

$$(4p_1 - 3)x_1 + x_2 + (1 - 4p_1)x_3 + x_4 = 0 \tag{3.1}$$

and

$$(4p_1 - 3)y_1 + y_2 + (1 - 4p_1)y_3 + y_4 = 0 \tag{3.2}$$

Without loss of generality we may state that $A(-a, 0), B(0, b), C(x_3, y_3)$ and $D(0, d)$ with $a > 0$ and $d > b$. We have $A_1 = \frac{1}{2}a(d - b)$, $A_2 = \frac{1}{2}x_3(d - b)$ and $p_1 = \frac{a}{a+x_3}$.

When we solve the above equation (1) for x_3 we find $x_3 = \pm a$. The only solution that holds is $x_3 = a$. With the second equation we find $y_3 = b + d$.

So ABCD is a parallellogram

Lemma

The area A of a simple region R can be calculated with an integral over the boundary C of R.

$$A = \oint_C (x\,dx + x\,dy)$$

Proof: We use the Theorem of Green:

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

Suppose $Q(x, y) = P(x, y) = x$, then we simply get:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

So the righthand side is the area A of the region R. P_0, P_1, P_2, ... P_n, where $P_n = P_0$. The area of the polygon can be calculated by the circular integral over the sides.

$$A = \oint_C (x\,dx + x\,dy) = \sum_{i=0}^{n-1} I_i$$

Here is I_i the contibution to the integral along the line segment P_iP_{i+1}. P_i has coordinates (x_i, y_i).

Suppose $x_{i+1} \neq x_i$, than we find the following equation for the line through vertices P_i and P_{i+1}

$$y - y_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i)$$

so

$$dy = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}\, dx$$

Now we calculate the integral I_i.

$$\begin{aligned} I_i &= \int_{x_i}^{x_{i+1}} (x\,dx + x\,dy) = \int_{x_i}^{x_i+1} x\,dx + \int_{x_i}^{x_i+1} x\frac{y_{i+1} - y_i}{x_{i+1} - x_i}\, dx \\ &= \left(1 + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right) \int_{x_i}^{x_{i+1}} dx \\ &= \left(1 + \frac{y_{i+1} - y_i}{x_i + 1 - x_i}\right) \cdot \frac{1}{2}({x_{i+1}}^2 - {x_i}^2) \\ &= \frac{1}{2}({x_{i+1}}^2 - {x_i}^2) + \frac{1}{2}(y_{i+1} - y_i)(x_{i+1} + x_i) \end{aligned}$$

We easily see that we may use the same formula for the case $x_{i+1} = x_i$

We now calculate $A = \sum_{i=0}^{n-1} I_i$. From the fact that $x_n = x_0$ follows immediately that quadratic terms cancel out. Conclusion:

$$A = \sum_{i=0}^{n-1} \frac{1}{2}(y_{i+1} - y_i)(x_{i+1} + x_i)$$

So

$$A = \frac{1}{2}\sum_{i=0}^{n-1}(y_{i+1} - y_i)(x_{i+1} + x_i) \qquad (3.3)$$

Example

The triangle with vertices (0,0), (4,3) and (3,5) has an area

$$A = \frac{1}{2}((3-0)(0+4) + (5-3)(4+3) + (0-5)(3+0)) = \frac{12 + 14 - 15}{2} = \frac{11}{2}.$$

Which can be verified by elementary means.

4
Opgave A NAW 5/4 nr. 4, December 2003

The problem

Introduction

For each non-negative integer n, let a_n be the number of digits in the decimal expansion of 2^n that are at least 5. Evaluate the sum $\sum_{n=0}^{\infty} \frac{a_n}{2^n}$.

Figure 4.1: NAW December 2003

Solution

Let $b(n)$ be the number of odd digits in the decimal expansion of 2^n. We can easily see that (change of notation) $a(n) = b(n+1)$, because a digit with value 5 or higher in 2^n generates an odd digit in the next generation 2^{n+1}. The sequence $b(n)$ is well known from Sloane's On-Line Encyclopedia of Integer Sequences as $A055254$. See [1] and [2]. We evaluate

$$\sum_{n=0}^{\infty} \frac{a(n)}{2^n} = \sum_{n=0}^{\infty} \frac{b(n+1)}{2^n}$$

We do not know a formula for $a(n)$ nor $b(n)$ other than an algorithm that can be implemented for instance in Maple.

```
A055254:=proc(n) local i, j, k, val;
val:= 2^n; j:=0; k:= floor(ln(val)/ln(10))+1;
for i from 1 to k do
   if (val mod 10) mod 2 = 1 then j:=j+1 fi;
```

```
   val:=floor(val/10);
od;
RETURN(j);
end:
```

When we calculate the sum of the first 200 terms we get:

$$\frac{357097343168664505675991576075813911671600665285065074511387}{1606938044258990275541962092341162602522202993782792835301376}$$

and approximately

$$0.2222222222222222222222222222222222222222222222222222222222220362$$

With 1000 terms we even get more 2s in the decimal expansion, so there is circumstantial evidence enough to evaluate the sum in question to $\frac{2}{9}$.

This is not much of a proof. Is there a kind of CAS-equivalent of 'epsilontics': For $\epsilon > 0$ there is a N_ϵ, such that $n > N_\epsilon$ implies $|\frac{2}{9} - \sum_{k=0}^{n} a(k)/2^k| < \epsilon$?

Not every ϵ can be represented in a computer system.

See https://www.nieuwarchief.nl/serie5/pdf/naw5-2004-05-2-174.pdf for a real proof.

Figure 4.2: NAW June 2004

References

[1] Neil Sloane, The On-Line Encyclopedia of Integer Sequences, Notices of the AMS, Vol. 50 nr 8 (September 2003), 912-915.

[2] https://oeis.org

[3] https://www.maplesoft.com/

5
Opgave B NAW 5/4 nr. 4, December 2003

The problem

Introduction.

Let G be a group such that squares commute and cubes commute,
i.e., $g^2h^2 = h^2g^2$ and $g^3h^3 = h^3g^3$ for all $g, h \in G$.

Show that G is Abelian.

Figure 5.1: NAW December 2003

Solution.

We define $(x, y) = x^{-1}y^{-1}xy$, called commutator of x and y. From this definition follows $(x, y) = 1$ if, and only if, $xy = yx$. Thus all commutators in group G are 1 if, and only if, G is an Abelian group. The subgroup G' of G generated by all commutators (x, y) is called the commutator subgroup or derived group. The factor group G/G' is Abelian.

Our problem can be translated in the statement: The factor group G/K is Abelian if K is the group generated by the commutators (x^2, y^2) and (x^3, y^3) with x and y in G. Or $(x^2, y^2) = 1 \wedge (x^3, y^3) = 1$ implies $(x, y) = 1$.

Let G be the free group generated by $'a'$ and $'b'$. Can we proof that the factor group $G/[(a^2, b^2), (a^3, b^3)]$ is Abelian?

See https://www.nieuwarchief.nl/serie5/pdf/naw5-2004-05-2-174.pdf for a real proof.

Figure 5.2: NAW June 2004

Remark.

This kind of problem is not in the educational vein. The solution just shows you how to manipulate some expressions! It is nothing more than a trick.

6

Opgave C NAW 5/4 nr. 4, December 2003

The problem

Introduction.

Let $(X_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables with $P\{X_n = 1\} = P\{X_n = -1\} = \frac{1}{2}$. Set $S_n = \sum_{k=1}^{n} X_k$. Calculate $P\{S_3 = 1 \vee S_6 = 2 \vee \cdots \vee S_{3n} = n \vee \cdots\}$.

Solution.

Let $\mathcal{P}(n) = P\{S_3 = 1 \vee S_6 = 2 \vee \cdots \vee S_{3n} = n\}$ and $A_n = \{1, 2, 3, ..., n\}$. We notice that

$$P\{S_{3k} = k\} = \frac{\binom{3k}{k}}{2^{3k}}$$

With the principle of inclusion/exclusion we get

$$\mathcal{P}(n) = P(n,1) - P(n,2) + \cdots + (-1)^{k-1}P(n,k) + \cdots (-1)^{n-1}P(n,n)$$

where

$$\begin{aligned} P(n,k) &= \sum_{\{i_1,i_2,\cdots,i_k\}\subset A_n} P\{S_{3i_1} = i_1 \vee S_{3i_2} = i_2 \vee \cdots \vee S_{3i_k} = i_k\} = \\ &= \sum_{i_1<i_2<\cdots<i_k\leq n} \frac{\binom{3i_1}{i_1}\binom{3i_2-3i_1}{i_2-i_1}\cdots\binom{3i_k-3i_{k-1}}{i_k-i_{k-1}}}{2^{3i_1}2^{3i_2-3i_1}\cdots 2^{3i_k-3i_{k-1}}} = \\ &= \sum_{i_1<i_2<\cdots<i_k\leq n} \frac{\binom{3i_1}{i_1}\binom{3i_2-3i_1}{i_2-i_1}\cdots\binom{3i_k-3i_{k-1}}{i_k-i_{k-1}}}{2^{3i_k}} \end{aligned}$$

We have to calculate $\lim_{n\to\infty} \mathcal{P}(n)$.

We see that

$$\begin{aligned} P(n+1,k) &= \sum_{i_1<i_2<\cdots<i_k\leq n+1} \frac{\binom{3i_1}{i_1}\binom{3i_2-3i_1}{i_2-i_1}\cdots\binom{3i_k-3i_{k-1}}{i_k-i_{k-1}}}{2^{3i_k}} = \\ &= P(n,k) + \sum_{i_1<i_2<\cdots<i_{k-1}<i_k=n+1} \frac{\binom{3i_1}{i_1}\binom{3i_2-3i_1}{i_2-i_1}\cdots\binom{3n+3-3i_{k-1}}{n+1-i_{k-1}}}{2^{3n+3}} \end{aligned}$$

and so

$$\mathcal{P}(n+1) = \mathcal{P}(n) + \mathcal{D}(n)$$

with

$$\mathcal{D}(n) = \sum_{k=1}^{n+1}(-1)^{k-1} \sum_{i_1<i_2<\cdots<i_{k-1}<i_k=n+1} \frac{\binom{3i_1}{i_1}\binom{3i_2-3i_1}{i_2-i_1}\cdots\binom{3n+3-3i_{k-1}}{n+1-i_{k-1}}}{2^{3n+3}}$$

We have $\mathcal{P}(2) = \mathcal{P}(1) + \mathcal{D}(1)$ and $\mathcal{P}(3) = \mathcal{P}(2) + \mathcal{D}(2) = \mathcal{P}(1) + \mathcal{D}(1) + \mathcal{D}(2)$, etcetera. Hence

$$\mathcal{P}(n) = \mathcal{P}(1) + \sum_{i=1}^{n-1}\mathcal{D}(i) = \frac{3}{8} + \sum_{i=1}^{n-1}\mathcal{D}(i)$$

Elementary counting gives the following results:

$$\begin{aligned} \mathcal{D}(1) &= 6/64 = 0.093750 \\ \mathcal{D}(2) &= 21/512 = 0.041016 \\ \mathcal{D}(3) &= 90/4096 = 0.021973 \\ \mathcal{D}(4) &= 429/32768 = 0.013092 \\ \mathcal{D}(5) &= 2184/262144 = 0.008331 \\ \mathcal{D}(6) &= 11628/2097152 = 0.005545 \\ \mathcal{D}(7) &= 63954/16777216 = 0.003812 \\ \mathcal{D}(8) &= 360525/134217728 = 0.002686 \\ \mathcal{D}(9) &= 2072070/1073741824 = 0.001930 \\ \mathcal{D}(10) &= 12096045/8589934592 = 0.001408 \end{aligned}$$

Total $\sum_{i=1}^{10}\mathcal{D}(i) = 1662515613/8589934592$, so

$$\mathcal{P}(11) = \frac{3}{8} + \sum_{i=1}^{10}\mathcal{D}(i) = 4883741085/8589934592 = 0.5685422901$$

We can do better: the sequence $a(n)_{n\geq 1} = 6, 21, 90, 429, 2184, 11628, \cdots$ can be written as:

$$a(n) = \frac{2}{3n+2}\binom{3n+3}{n+1}$$

and hence

$$\mathcal{D}(n) = \frac{a(n)}{2^{3n+3}}$$

Further we can write

$$\mathcal{P}(n) = \frac{3}{8} + \sum_{i=1}^{n-1} \mathcal{D}(i) = \frac{3}{8} + \sum_{i=1}^{n-1} \frac{\binom{3i+3}{i+1}}{(3i+2)2^{3i+2}} \qquad (6.1)$$

The probability in question is

$$\lim_{n\to\infty} \mathcal{P}(n) = \frac{3}{8} + \sum_{i=1}^{\infty} \frac{\binom{3i+3}{i+1}}{(3i+2)2^{3i+2}} = 0.57294901687515772769311\cdots$$

Conclusion.

The above calculations are based on a lemma:

$$\mathcal{D}(n) = \frac{2}{3n+2} P\{S_{3n+3} = n+1\} \qquad (6.2)$$

This lemma can be proved with induction on n, proving

$$a(n) = 2^{3n+3} \cdot \mathcal{D}(n) = \frac{2}{3n+2}\binom{3n+3}{n+1} = \frac{3(3n+1)}{(2n+1)(n+1)}\binom{3n}{n}$$

The summand

$$\mathcal{D}(i) = \frac{\binom{3i+3}{i+1}}{(3i+2)2^{3i+2}}$$

is a hypergeometric term, but not 'Gosperable', so there is no closed form for $\mathcal{P}(n)$ in the sense of [1][1] Definition 8.1.1. See [1] and [2]. Maple 8 gives a ${}_3F_2$ hypergeometric form.

$$\lim_{n\to\infty} \mathcal{P}(n) = \frac{3}{8} + \sum_{i=1}^{\infty} \frac{\binom{3i+3}{i+1}}{(3i+2)2^{3i+2}} = \frac{3}{8} + \frac{3}{32} \cdot {}_3F_2(1, \frac{5}{3}, \frac{7}{3}; \frac{5}{2}, 3; \frac{27}{32})$$

which evaluates to $0.57294901687515772769311\cdots$.

[1] Marko Petkovšek, Herbert Wilf, and Doron Zeilberger. *A = B*. A K Peters/CRC Press, Natick, MA, USA, 1996. ISBN 1568810636

References

[1] Petkovšek, Wilf and Zeilberger, $A = B$, A.K. Peters, Massachusetts, 1996.

[2] The Maple SumTools[Hypergeometric] library

7
Opgave A NAW 5/5 nr. 1, March 2004

The problem

Introduction

For every integer $n > 2$ prove that

$$\sum_{j=1}^{n-1} \left(\frac{1}{n-j} \sum_{k=j}^{n-1} \frac{1}{k} \right) < \frac{\pi^2}{6}$$

Solution

Let

$$s_{n-1} = \sum_{j=1}^{n-1} \left(\frac{1}{n-j} \sum_{k=j}^{n-1} \frac{1}{k} \right) \tag{7.1}$$

We have $s_1 = 1$, $s_2 = 1\frac{1}{4}$, $s_3 = \frac{49}{36} = \frac{5}{4} + \frac{1}{9}$ and $s_4 = s_3 + \frac{1}{4^2}$.

We shall prove the following

Proposition

$$s_n = s_{n-1} + \frac{1}{n^2} \quad \text{for} \quad n > 1 \tag{7.2}$$

From this proposition follows:

$$s_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} = \sum_{k=1}^{n} \frac{1}{k^2} \tag{7.3}$$

And we are finished, because $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6}$, we have $s_{n-1} < s_n < \frac{\pi^2}{6}$. *Note: this is true for* $n \geq 2$.

Now we prove the proposition.

Let $A_{n-1} = (a_{ij}) = (\frac{1}{n-i} \cdot \frac{1}{j})$ with $1 \leq i \leq j \leq n-1$ and $B_n = (b_{ij}) = (\frac{1}{n+1-i} \cdot \frac{1}{j})$ with $1 \leq i \leq j \leq n$.

Then

$$s_{n-1} = \sum_{1 \leq i \leq j \leq n-1} a_{ij} \quad \text{and} \quad s_n = \sum_{1 \leq i \leq j \leq n} b_{ij}$$

Comparing a_{ij} with b_{ij} we see $b_{ij} = a_{i-1,j}$ for $2 \leq i \leq j \leq n-1$.

So

$$s_n = s_{n-1} - \sum_{i=1}^{n-1} a_{ii} + \sum_{j=1}^{n} b_{1j} + \sum_{i=1}^{n} b_{in} - b_{nn}$$

We can write $a_{ii} = \frac{1}{(n-i)i} = \frac{1}{n(n-i)} + \frac{1}{ni}$, so

$$\sum_{i=1}^{n-1} a_{ii} = \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{n-i} + \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{i} = \frac{1}{n} H_{n-1} + \frac{1}{n} H_{n-1} = \frac{2}{n} H_{n-1}$$

where H_{n-1} is the $(n-1)$-th harmonic number.

See for instance Graham, Knuth, Patashnik, Concrete Mathematics, p. 258, section 6.3.

Further we know

$$\sum_{j=1}^{n} b_{1j} = \sum_{i=1}^{n} b_{in} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{n} H_n$$

Now

$$s_n = s_{n-1} - \frac{2}{n} H_{n-1} + \frac{1}{n} H_n + \frac{1}{n} H_n - \frac{1}{n^2}$$

and hence

$$s_n = s_{n-1} + \frac{2}{n}(H_n - H_{n-1}) - \frac{1}{n^2} = s_{n-1} + \frac{2}{n} \cdot \frac{1}{n} - \frac{1}{n^2}$$

This concludes the proof of the proposition

$$s_n = s_{n-1} + \frac{1}{n^2} \tag{7.4}$$

8

Opgave B NAW 5/5 nr. 1, March 2004

The problem

Introduction

Consider the first digit in the decimal expansion of 2^n for $n \geq 0$: 1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, $\cdots$. Does the digit 7 appear in this sequence? Which digit appears more often, 7 or 8? How many times more often?

Solution

The first question is easily solved affirmative: $2^{46} = 70368744177664$.

The sequence is well known from Sloane's On-Line Encyclopedia of Integer Sequences as $A008952$. See [1]. We use the formula

https://oeis/A008952

$$a(n) = \lfloor 2^n / 10^{\lfloor n \cdot \frac{\ln 2}{\ln 10} \rfloor} \rfloor$$

in a Maple Program [2] to calculate the frequency of digit d in all $a(k)$ with $k \leq n$.

```
> A:=proc(n,d) local i,k,s;
   s:=0;
   for k from 0 to n do
    i:=floor(2^k / 10^floor(k*ln(2)/ln(10)));
    if i=d then s:=s+1 fi;
   od;
   RETURN(s);
  end;
```

```
> seq(A(500,i),i=1..9);
   151, 88, 63, 49, 39, 34, 28, 26, 23
```

Frequency count $f(d)$ for leading digit d of 2^n:

$\leq n$	d	1	2	3	4	5	6	7	8	9
85		26	16	10	9	7	5	4	5	4
86		26	16	10	9	7	5	5	5	4
100		31	17	13	10	7	7	6	5	5
200		61	36	24	20	16	13	11	11	9
1000		302	176	125	97	79	69	56	52	45
2000		603	354	248	194	160	134	114	105	89
3000		904	529	374	291	238	201	173	155	136
4000		1205	705	499	388	317	269	230	207	181
5000		1506	882	623	485	397	335	288	259	226
6000		1807	1058	748	582	476	401	347	309	273
7000		2108	1233	874	679	554	468	406	359	320
8000		2409	1409	999	776	633	537	462	412	364
9000		2710	1587	1122	873	714	602	520	463	410
10000		3011	1761	1249	970	791	670	579	512	458
Benford's law	:	3010	1761	1249	969	791	669	579	511	457

The second question can be answered by: *'the winner is* 7'! From $n > 209$ or some more, the frequency of the digit 7 is greater than that of 8.

We can generalize: for n large enough the frequencies are a decreasing sequence, meaning for digits d_1 and d_2: $d_1 < d_2$ implies $f(d_1) > f(d_2)$. We can think of a reason: multiplication of 2^n with leading digit 1 with $2^{10} = 1024$, gives more often the same leading digit, compared with the larger leading digits 2, 3, $\cdots$, 9, and so on.

But Benford's law comes to the rescue (see [3]), our sequence is a well known example:

$$\text{Prob(first significant digit} = d) \ = log_{10}(1 + \frac{1}{d}), \text{ for } d = 1, 2, ..., 9$$

The similarity of the last two lines in the table above is striking!

The last question is the most difficult to answer. Our best guess for large n is according to Benford's law:

$$\frac{f(7)}{f(8)} \quad \text{tends to} \quad \frac{\log_{10}(1+1/7)}{\log_{10}(1+1/8)} = 1.133706496$$

References

[1] https://oeis.org

[2] https://www.maplesoft.com/

[3] Hill, T.P., The Significant-Digit Phenomenon, Amer. Math. Monthly 102, 322-327, 1995

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Opgave A NAW 5/5 nr. 2, June 2004

The problem

Introduction

The sequence 3331113331313331113333... is identical to the sequence of its block lengths. Compute the frequency of the number 3 in this sequence.

Solution

This sequence is known as the Kolakoski-(3,1) sequence. See Neil Sloane's On-Line Encyclopedia of Integer Sequences, sequence number *A*064353, which is in fact the Kolakoski-(1,3) sequence, different only in the first position. See [1].

https://oeis/A064353

Michael Baake and Bernd Sing wrote: Unlike the (classical) Kolakoski sequence on the alphabet {1,2}, its analogue on {1,3} can be related to a primitive substitution rule. See [2] and [3]. We base our calculations on section 2 of this paper.

Let $A = 33$, $B = 31$ and $C = 11$. In the case of $\mathrm{Kol}(3,1)$ the substitution σ and the matrix M of the substitution are given by

$$\sigma : \begin{array}{ccc} A & \mapsto & ABC \\ B & \mapsto & AB \\ C & \mapsto & B \end{array} \quad \text{and} \quad M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (9.1)$$

where $m_{ij} = 1$ if and only if there is corresponding mapping in σ, for instance $A \mapsto ABC$ corresponds to the fist column of M, etcetera.

An infinite fixed point can be obtained as follows:

$$A \mapsto ABC \mapsto ABCABB \mapsto \ldots \tag{9.2}$$

This corresponds to

$$333111333131\ldots \tag{9.3}$$

which is the unique infinite $\mathrm{Kol}(3,1)$. The matrix M is primitive because M^3 has only positive entries. The characteristic polynomial $P(\lambda)$ of M is

$$P(\lambda) = \lambda^3 - 2\,\lambda^2 - 1, \tag{9.4}$$

and has one real root λ_1 and two complex roots $\lambda_{2,3}$. We have

$$2.205569 \approx \lambda_1 > 1 > |\lambda_2| = |\lambda_3| \approx 0.67 \tag{9.5}$$

According to the Perron-Frobenius Theorem there is a positive eigenvector to λ_1. We easily verify that $\mathbf{x_1} = (\lambda_1, \lambda_1^2 - \lambda_1, 1)^T$ is such an eigenvector.

See wikipedia

Figure 9.1: Wikipedia Page

Starting with $\mathbf{x}(0) = (1,0,0)^T$ we define

$$\mathbf{x}(k+1) = M\mathbf{x}(k) \tag{9.6}$$

The asymptotical behavior of this system will be of the form $\mathbf{x}(n) = c \cdot (\lambda_1)^n \mathbf{x_1}$ for some value of c.

From $\mathbf{x}(n)$ we can calculate the number of A's, B's and C's. In $A = 33$ there are two 3's, etcetera, so we can easily calculate the relative frequencies of the letters of the alphabet. The frequency of the '3':

$$\rho_3 = \frac{2 \cdot \lambda_1 + 1 \cdot (\lambda_1^2 - \lambda_1) + 0 \cdot 1}{2 \cdot (\lambda_1^2 + 1)} \approx 0.6027847150 \tag{9.7}$$

References

[1] https://oeis.org/A064353

[2] Baake, Sing: Kolakoski-(3,1) is a (deformed) Model Set, Canad. Math. Bull. 47, No. 2, 168–190 (2004)

[3] See also https://arxiv.org/abs/math.MG/0206098

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Opgave C NAW 5/5 nr. 2, June 2004

The problem

Introduction

Let A be a ring and let $B \subset A$ be a subring. As a subgroup, B has finite index in A. Show that there exists a two-sided ideal I of A such that $I \subset B$ and I has finite index as a subgroup of A.

Solution

We have A and B as defined above. The index $[A : B] = k < \infty$ or with other words, the additive factor group A/B is a finite Abelian group build from cosets of type $x + B$.

Let $\mathcal{E}(G)$ be the ring of endomorphisms of the Abelian group G. We define a ring homomorphism f: $B \to \mathcal{E}(A/B)$: for $a \in B$ we define f: $a \mapsto \alpha$ with $(x + B)\alpha = xa + B$. Note that we use here the *right* function notation, avoiding the notion of anti-homomorphism (see [2]).

The kernel of f is $L = \{a \in B | Aa \subset B\}$, L is the largest left-ideal of A with $L \subset B$. The factor group B/L is isomorphic to a subgroup of $\mathcal{E}(A/B)$, so B/L is a finite Abelian group and since $(A/L)/(B/L) \cong A/B$ it follows that A/L is finite Abelian.

We now consider the ring homomorphism g: $A \to \mathcal{E}(A/L)$: for $b \in A$ we define g: $b \mapsto \beta$ with $\beta(x + L) = bx + L$. Its restriction to L, g_L: $L \to \mathcal{E}(A/L)$ has kernel

$$I = \{a \in L | aA \subset L\} = \{a \in B | Aa \subset B \wedge aA \subset B\}.$$

I is the largest two-sided ideal of A with $I \subset B$. We have L/I finite and hence A/I is a finite Abelian group, so $[A : I] < \infty$.

References

[1][1] Marshall Hall, Jr. The Theory of Groups, Macmillan, New York, 1959.

[2] https://planetmath.org/encyclopedia/UnitalModule.html

Look up this URL in the Wayback Machine. It may or may not help you!

[1] Jr Hall, Marshall. *The Theory of Groups*. The Macmillan Company, New York, 1959

11

Problem A NAW 5/5 nr. 4, December 2004

The problem

Introduction

1. Show that there exist infinitely many $n \in \mathbf{N}$, such that $S_n = 1 + 2 + ... + n$ is a square.

2. Let $a_1, a_2, a_3, ...$ be those squares. Calculate $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution

We know $S_n = \frac{1}{2}n(n+1)$ so we have to solve the diophantine equation

$$\frac{1}{2}n(n+1) = m^2 \tag{11.1}$$

Rewriting gives $4n^2 + 4n = 8m^2$ or $(2n+1)^2 - 1 = 2(2m)^2$. Substituting $x = 2n+1$ and $y = 2m$ we get the Pell equation

$$x^2 - 2y^2 = 1 \tag{11.2}$$

with an infinite number of solutions $(3,2), (17,12), (99,70)...$ with corresponding $n = 1, 8, 49, 288, ...$

A well known result gives solutions of (2)

$$x_k = \frac{(3+2\sqrt{2})^k + (3-2\sqrt{2})^k}{2}$$

and

$$y_k = \frac{(3+2\sqrt{2})^k - (3-2\sqrt{2})^k}{2\sqrt{2}}$$

The sequence $\{a_i\}_{i=1,2,3,...}$ starts with $1, 36, 1225, 41616, ...$ and can be calculated with

$$a_k = \frac{y_k^2}{4} = \frac{((3+2\sqrt{2})^k - (3-2\sqrt{2})^k)^2}{32} \tag{11.3}$$

which can be rewritten as

$$a_k = \frac{((17+12\sqrt{2})^k + (17-12\sqrt{2})^k) - 2}{32} \tag{11.4}$$

And so

$$\frac{a_{k+1}}{a_k} = \frac{((17+12\sqrt{2})^{k+1} + (17-12\sqrt{2})^{k+1}) - 2}{((17+12\sqrt{2})^k + (17-12\sqrt{2})^k) - 2} \tag{11.5}$$

We easily see that $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 17 + 12\sqrt{2}$.

Remark

Finding a triangular number S_n that is cubic, except the trivial 1, would be spectacular. As we try to solve

$$\frac{1}{2}n(n+1) = m^3$$

substituting $X = 2m$ and $Y = 2n + 1$ we get the elliptic curve with equation

$$Y^2 = X^3 + 1 \tag{11.6}$$

We find this curve as A36 in the Cremona table. The torsion group is of order 6 with real members (-1,0), (0,-1), (0,1), (2,-3) and (2,3). This means the only cubic triangular number is 1.

Moreover the above equation is also known from the Catalan conjecture, or should we say Catalan theorem: the only non-trivial integer powers that differ 1 are 2^3 and 3^2.

References

[1] https://oeis.org/A001110

12

Problem B NAW 5/5 nr. 4, December 2004

The problem

Introduction.

Let G be a finite set of elements and $\cdot$ a binary associative operation on G. There is a neutral element in G and that is the only element in G with the property $a \cdot a = a$.

Show that G with the operation $\cdot$ is a group.

Solution.

G is a finite semigroup with identity. Let A be a subset of G. There is a smallest subsemigroup K of G which contains A. We say A generates K, notation $\{A\} = K$. A single element x of G generates a subsemigroup $\{x\} = \{x^n | n > 0\}$. Since $\{x\}$ is finite there must be integers $p > q$, such that $x^p = x^q$. So $x^p = x^{q+k} = x^q x^k = x^k x^q = x^q$ and $e = x^k$ is a neutral element for $\{x\}$. We assume that k is the smallest integer with this property.

We easily verify that $\{x\} = \{e, x, x^2, ..., x^{k-1}\}$ is a group with neutral element e and as such a subgroup of G. Clearly e is idempotent with $e \cdot e = e^2 = e$. According to the problem statement e is the only element of G with this property.

We now proof the following lemma:

Let G be a finitely generated semigroup and H een subgroup of G. Then there exists a maximal subgroup M of G containing H.

Proof: Let G be generated by $x_1, ..., x_m$ and let y_1 be the first of the x_i not contained in H and with property $H_1 = \{H, y_1\}$ is a group. If such a y_1 does not exist then $M = H$ is the maximal subgroup of G. We now have $H_1 \supseteq H$. If $H_1 = G$, then G is the maximal subgroup sought. If not, choose $H_2 = \{H_1, y_2\} \supseteq H_1$, where y_2 is the first of the x_i not contained in H_1 and $\{H_1, y_2\}$ is a group. If such a y_2 does not exist then $M = H_1$ is the maximal subgroup of G.

Continuing this proces we must reach the situation where no more extension is possible: $H_i \supseteq H_{i-1} \supseteq ... \supseteq H$, H_i is a group. If $H_i = \{H_{i-1}, y_i\} = G$ the maximal subgroup is G else the maximal subgroup $M = H_i$ is a proper subgroup of G.

G is finite and so certainly finitely generated. According to the above lemma $\{x\}$ is contained in a maximal subgroup M. If $M = G$ we are ready, but let there be a y not in M, then $\{y\}$ is contained in a maximal subgroup M', with neutral element e', with $e' \cdot e' = e'$. If $e' \neq e$ we have a contradiction and there is no such element y, hence $M = G$. If $e' = e$ than we easily see that $\{M, y\}$ is a group in contradiction with the maximality of M. So we have proved that G is a group.

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Problem A NAW 5/6 nr. 1, March 2005

The problem

Introduction

Calculate

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} i^2}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} i^3}$$

Solution

This kind of problems make me feel young. They remind me to the early sixtees and the lectures of Prof. Van der Blij.

Figure 13.1: Wikipedia

Part 1

By a well known result we first write $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ and hence the first summand can be written as

$$\frac{6}{n(n+1)(2n+1)} = \frac{6}{n} + \frac{6}{n+1} - \frac{24}{2n+1}$$

Let

$$S_n = \sum_{k=1}^{n} \frac{1}{\sum_{i=1}^{k} i^2} = 6\sum_{k=1}^{n} \frac{1}{k} + 6\sum_{k=1}^{n} \frac{1}{k+1} - 24\sum_{k=1}^{n} \frac{1}{2k+1}$$

so using a result on harmonic numbers we get

$$S_n = 6H_n + 6(H_n - 1) - 24(H_{2n+1} - \frac{1}{2}H_n - 1) = 18 - 24(H_{2n+1} - H_n)$$

H_n being the n-th harmonic number. We know $H_n = \ln n + \Delta_n$ with $\lim_{n\to\infty} \Delta_n = \gamma$, Euler's constant.

Now with $H_{2n+1} - H_n = \ln(2n+1) - \ln n - \Delta_{2n+1} + \Delta_n$ we can easily see that $\lim_{n\to\infty}(H_{2n+1} - H_n) = \ln 2$ and therefor the first answer is $18 - 24\ln 2$.

Part 2

First we write $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ and hence the second summand can be written as

$$\frac{4}{n^2(n+1)^2} = \frac{4}{(k+1)^2} + \frac{4}{k^2} - \frac{8}{k} + \frac{8}{k+1})$$

Let

$$S_n = \sum_{k=1}^{n} \frac{1}{\sum_{i=1}^{k} i^3} = \sum_{k=1}^{n} (\frac{4}{(k+1)^2} + \frac{4}{k^2} - \frac{8}{k} + \frac{8}{k+1})$$

so

$$S_n = 4\sum_{k=1}^{n} \frac{1}{(k+1)^2} + 4\sum_{k=1}^{n} \frac{1}{k^2} - 8\sum_{k=1}^{n} \frac{1}{k} + 8\sum_{k=1}^{n} \frac{1}{k+1}$$

and

$$S_n = 4(\sum_{k=1}^{n} \frac{1}{k^2} - 1) + 4\sum_{k=1}^{n} \frac{1}{k^2} - 8H_n + 8(H_n - 1) = 8\sum_{k=1}^{n} \frac{1}{k^2} - 12$$

So

$$\lim_{n\to\infty} S_n = 8 \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} - 12 = 8 \cdot \frac{1}{6}\pi^2 - 12 = \frac{4}{3}\pi^2 - 12$$

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Problem C NAW 5/6 nr. 1, March 2005

The problem

Introduction

We call a triangle integral if the sides of the triangle are integral. Consider the integral triangles with rational circumradius.

1. Prove that for any positive integral p there are only a finitely many integrals q such that there exists an integral triangle with circumradius equal to $\frac{p}{q}$.
2. Prove that for any positive integral q there exist infinitely many integral triangles with circumradius equal to $\frac{p}{q}$ for an integral p with $\gcd(p,q) = 1$.

Solution

Let triangle ABC have integral sides a, b and c with area A and circumradius R. There exists a relation between this quantities given by Heron's formulae

$$(4A)^2 = (a+b+c)(a+b-c)(a-b+c)(-a+b+c) \qquad (14.1)$$

and

$$A = \frac{abc}{4R} \qquad \text{or equivalently} \qquad R = \frac{abc}{4A} \qquad (14.2)$$

So there is an one-to-one relation between the set of all integral triangles with rational/integral area and the set of all integral triangles with rational circumradius. The set of all integral triangles with integral area is well studied as the Heronian triangles.

Part 1

The sides being integral implies the existance of a minimal circumradius R_{min}. For a given p there exist only finitely many q with $\frac{p}{q} > R_{min}$. So there are only finitely many q with $R = \frac{p}{q}$ the circumradius of an integral triangle.

Part 2

The case $q = 1$ is trivial, as it is a well known fact that there are infinitely many numbers p where p is the hypothenuse of a Pythagorean triangle. Scaling by two gives an integral triangle with circumradius $R = p$.

For the case $q > 1$ we use a parametric representation of the Heronian triangles as found in [1]

$$a = n(m^2 + k^2) \quad (14.3)$$
$$b = m(n^2 + k^2) \quad (14.4)$$
$$c = (m + n)(mn - k^2) \quad (14.5)$$
$$A = kmn(m + n)(mn - k^2) \quad (14.6)$$

For any integers m, n and k with $mn > k^2 > \frac{m^2 n}{(2m+n)}$, $\gcd(m, n, k) = 1$ and $m \geq n \geq 1$ we have one member of each simularity class of the Heronian triangles.

Using this and (2) we get

$$R = \frac{(m^2 + k^2)(n^2 + k^2)}{4k} \quad (14.7)$$

In our case we do not need the restriction to unique reduced Heronian triangles. For the problem at hand we only need the triangle inequalities $a + b > c$, $a + c > b$ and $b + c > a$, together with $mn - k^2 > 0$. As we can easily see this can be realised by $m > k$, $n > k$ and $k \geq 1$.

Let $k = q$ and $p = \frac{(m^2+q^2)(n^2+q^2)}{4}$. All we have to prove is the existance of infinitely many (m, n) such that 4 is a divisor of $(m^2 + q^2)(n^2 + q^2)$. If q is even than choose $n > q$ with $\gcd(n, q) = 2$ so $4|(n^2 + q^2)$, let $m > q$ be a positive integer with

$\gcd(m, q) = 1$. If q is not even choose $n > q$ and $m > q$ both not even with $\gcd(n, q) = \gcd(m, q) = 1$, so $2|(n^2 + q^2)$ and $2|(m^2 + q^2)$.

The sums $m^2 + q^2$ and $n^2 + q^2$ or $(\frac{n}{2})^2 + (\frac{q}{2})^2$ are so called primitive sums of two squares, defined by $x^2 + y^2$ with $\gcd(x, y) = 1$. For a prime divisor of such a primitive sum $x^2 + y^2$ it is not possible to be a divisor of y. In all cases we easily verify that we have a Heronian triangle with circumradius $R = \frac{p}{q}$ with $\gcd(p, q) = 1$, so we have infinitely many of them.

Reference

[1] Buchholz, R. H., Perfect Pyramids, Bull. Austral. Math. Soc. 45, nr 3, 1992.

15

Problem A NAW 5/6 nr. 2, June 2005

The problem

Introduction

A student association organises a large-scale dinner for 128 students. The chairs are numbered 1 through 128. The students are also assigned a number between 1 and 128. As the students come into the room one by one, they must sit at their assigned seat. However, 1 of the students is so drunk that he can't find his seat and takes an arbitrary one. Any sober student who comes in and finds his seat taken also takes an arbitrary one. The drunken student is one of the first 64 students. What is the probability that the last student gets to sit in the chair assigned to him?

Solution

We solve this problem for n students with $n > 1$. Without loss of generality we may assume that the first student is drunk, see below. There are three possibilities: student 1 seats on seat 1 (we call this success, because student n will be seated on seat n), student 1 seats on seat n (failure) or student 1 seats on a remaining arbitrary seat k.

In the last case the next students with numbers less than k will be seated on their assigned seat. Student k will now act as a drunken student by taking an arbitrary free seat. So in a way student k becomes the new number '1' of a corresponding problem with $n - k + 1$ students. The choices are: Student '1' seats on seat 1 (success), he or she seats on the last seat (failure) or again the choice of an arbitrary free seat. This process is repeated until we have eventually two students, the 'first' and the last and there are only two choices, one leads to success, the other to failure.

In the end 'success' and 'failure' are completely symmetric in this story. In all possible stages of the process success and failure have the same probability, so the probability the last student will be seated on the last chair is 1/2.

See https://www.nieuwarchief.nl/serie5/pdf/naw5-2005-06-4-332.pdf

Figure 15.1: QR-code link

16

Problem C NAW 5/6 nr. 2, June 2005

The problem

Introduction

This problem plays a role in a few sequences in the OEIS: https://oeis.org/A111774, https://oeis.org/A111775, https://oeis.org/A111787, https://oeis.org/A109814 and https://oeis.org/A174090

In what follows, P stands for the set consisting of all odd prime numbers; M is the set consisting of all natural 2-powers $1, 2, 4, 8, 16, 32, ...$; T is the set consisting of all positive integers that can be written as a sum of at least three consecutive natural numbers.

1. Show that the set theoretic union of P, M and T coincides with the set consisting of all the natural numbers..
2. Show that the sets P, M and T are pairwise disjoint.
3. Given $b \in T$, determine $t(b)$ in terms of the prime decomposition of b, where by definition $t(b)$ stands for the minimum of all those numbers $t > 2$ for which b admits an expression as sum of t consecutive natural numbers.
4. Consider the cardinality $C(b)$ of the set of all odd positive divisors of some element b of T. Now think of expressing this b in all possible ways as a sum of at least three consecutive natural numbers. Suppose this can be done in $S(b)$ ways. Determine the numerical connection between the numbers $C(b)$ and $S(b)$.

Remark: In this problem we clearly follow the convention not to include zero in the natural numbers.

Solution

First let

$$a = p_0^{e_0} \cdot p_1^{e_1} \cdots p_m^{e_m} \tag{16.1}$$

be the 'prime decomposition' of a positive integer a with $p_0 = 2$, $e_0 \geq 0$ and $p_1, ..., p_m$ odd primes with $e_i > 0$ for $i = 1, ..., m$. We want to write a as the sum of k consecutive natural numbers starting with n.

$$a = n + (n+1) + \cdots (n+k-1) = k \cdot n + \frac{k(k-1)}{2} = k(2n+k-1)/2$$

So

$$'2a = k \cdot (2n + k - 1) \tag{16.2}$$

We define k to be the smallest factor, thus $k < \sqrt{2a}$. We observe that only one of the factors is odd.

Part 1 and 2

When a is a power of 2 we can only have $k = 1$. A power of two is clearly not an odd prime and vice versa. An odd prime can only be written as a sum of 2 consecutive natural numbers ($k = 2$). For all other positive integers we have at least one odd prime divisor p_i. Let $k = p_i \geq 3$ and $n = (2a/k - k + 1)/2$. It follows that a can be written as the sum of at least three consecutive positive integers starting with n. The rest is trivial.

Part 3

Let $b = a \in T$ and p_1 the smallest odd prime divisor of b. From (2) it follows that if $e_0 = 0$, meaning b is odd, we have $t(b) = p_1$, else $t(b) = min(2^{e_0+1}, p_1)$.

Part 4

Let again be $b = a \in T$. We use the prime decomposition (1) to find the number of all odd divisors of b. We easily see that this number must be

$(e_1+1)\cdot(e_2+1)\cdots(e_m+1)$. So $C(b) = (e_1+1)\cdot(e_2+1)\cdots(e_m+1)$.

$S(b)$ is the number of ways b can be expressed as sum of at least three positive integers.

From (2) it follows that for each odd divisor of b we can find a $k < \sqrt{2a}$. We must exclude $k = 1$ and $k = 2$. Only in case of an odd b we can have $k = 2$, so $S(b) = C(b) - 2$ if b is odd and $S(b) = C(b) - 1$ if b is even.

See https://www.nieuwarchief.nl/serie5/pdf/naw5-2005-06-4-332.pdf

Figure 16.1: QR-code link

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Problem B NAW 5/5 nr. 3, October 2005

The problem

Introduction.

1. Let G be a group and suppose that the maps $f, g : G \to G$ with $f(x) = x^3$ and $g(x) = x^5$ are both homomorphisms. Show that G is Abelian.

2. In the previous excercise, by which pairs (m, n) can $(3, 5)$ be replaced if we still want to be able to prove that G is Abelian.

Solution.

Part 1

Let $(ab)^5 = a^5b^5$ for all $a, b \in G$, then we easily see that $(ba)^4 = a^4b^4$. Now $(ab)^3 = a^3b^3$ for all $a, b \in G$ and hence $(ba)^2 = a^2b^2$. So $(a^2b^2)^2 = a^4b^4$ and $b^2a^2 = a^2b^2$. Hence in G squares commute.

Now $a^4b^4 = b^4a^4 = (ba)^4$ and so $b^3a^3 = (ab)^3 = a^3b^3$ and hence in G cubes commute. In the solution of Opgave 2003-4B from the UWC it is proved that in this case G is Abelian.

See Chapter 5

Part 2

We define $f_n(x) = x^n$ for $x \in G$. f_m and f_n are homomorphisms.

The case $m = 2$ $(m \leq n)$ is trivial because from $(ab)^2 = a^2b^2$ follows immediately $ba = ab$, etcetera.

From $(ab)^m = a^m b^m$ and $(ab)^n = a^n b^n$ $(m < n)$ follows $(ba)^{m-1} = a^{m-1}b^{m-1}$ and $(ba)^{n-1} = a^{n-1}b^{n-1}$.

If $k(m-1) = n-1$ and $m-1 = n-m$ we get $(a^{m-1}b^{m-1})^k = a^{n-1}b^{n-1}$. So $n = 2m-1$ and $k = 2$ and we may conclude that the $(m-1)$-th powers commute.

See https://www.nieuwarchief.nl/serie5/pdf/naw5-2006-07-1-066.pdf

Figure 17.1: QR-code link

See also

Vlastimil Dlab, A note on powers of a group, Acta Sci. Math. (Szeged) 25, 1964, pp. 177-178.

18

Problem C NAW 5/6 nr. 4, December 2005

The problem

Introduction

For a finite affine geometry there are a finite number of points and the axioms are as follows:

1. Given two distict points, there is exactly one line that includes both points.
2. The parallel postulate: Given a line L and a point P not on L, there exists exactly one line through P that is parallel to L.
3. There exists a set of four points, no three collinear.

We denote the set of points by **P**, and the set of lines by **L**. We define an automorphisme or collineation σ the usual way (a collineation keeps collinearity).

Prove that there exist a point $P \in \mathbf{P}$ with $\sigma(P) = P$ or a line $L \in \mathbf{L}$ with $\sigma(L) = L$ or $\sigma(L) \cap L = \emptyset$.

Solution

Let π be a finite affine plane of order n. π can be canonically embedded in a projective plane $\bar{\pi}$ of order n by adding a line L_∞ and a point on every line L of π: $L \wedge L_\infty$, where parallel lines L and L' share the same point on L_∞.

$\bar{\pi}$ has $n^2 + n + 1$ points P_i and an equal number of lines L_i. Let $N = n^2 + n + 1$. We define an incidence matrix $A = (a_{ij})$ of order N:

$$a_{ij} = 1 \quad \text{if} \quad P_i \in L_j \quad \text{and} \quad a_{ij} = 0 \quad \text{if} \quad P_i \notin L_j$$

We see that

$$AA^T = A^T A = nI + J \tag{18.1}$$

with J a matrix with every entry 1.

A collineation σ of π can be extended to a collineation of $\bar{\pi}$, also indicated by σ. σ acts on the points P_i as a permutation P and as a permutation Q on the lines L_i. We write P and Q as $(0,1)$-matrices of order N with entries:

$$p_{ij} = 1 \quad \text{if} \quad \sigma(P_i) = P_j$$
$$q_{ij} = 1 \quad \text{if} \quad \sigma(L_i) = L_j$$

and $p_{ij} = 0$, $q_{ij} = 0$ otherwise.

We now have

$$AQ = PA$$

and according to (1) we have

$$(det(A))^2 = det(nI + J) = (n+1)^2 n^{N-1} > 0$$

So

$$Q = A^{-1} P A$$

P and Q are similar as matrices, but also as permutations. Especially P and Q have the same number of cycles of length one, also called fixed "points".

$\sigma(L_\infty) = L_\infty$, so there must be at least one fixed point. If there are no fixed points on L_∞ there is a affine point P with $\sigma(P) = P$. If there is a fixed point on L_∞, say $L \wedge L_\infty$, then $\sigma(L) \parallel L$, meaning $\sigma(L) = L$ or $\sigma(L) \cap L = \emptyset$.

See https://www.nieuwarchief.nl/serie5/pdf/naw5-2006-07-2-147.pdf By mistake my name is not mentioned. I think my method is quite original.

Figure 18.1: QR-code link

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Problem B NAW 5/7 nr. 1, March 2006

The problem

Introduction

Let $P = (0,0)$ and $Q = (3,4)$. Find all points $T = (x, y)$ such that

- x and y are integers,
- the length of the line segments PT and QT are integers.

Solution

Let the length of the line segments PT and QT be denoted by $|PT|$ and $|QT|$. $|PT| - |QT|$ can take integer values ranging from -5 to 5. Let $||PT| - |QT|| = d$ with $d = 0, 1, 2, 3, 4, 5$. So

$$(\sqrt{(x^2 + y^2} - \sqrt{(x-3)^2 + (y-4)^2})^2 = d^2$$

and

$$(36 - 4d^2)x^2 + 96xy + (64 - 4d^2)y^2 + (-300 + 12d^2)x + (-400 + 16d^2)y + (d^2 - 25)^2 = 0$$

For $d = 0$ this simplifies to $y = \frac{-6x+25}{8}$ with clearly no integer solutions.

For $d = 5$ the equation simplifies to $4x - 3y = 0$ with solutions $(x, y) = (3k, 4k)$ with integer k.

For $d = 4$ the equation reduces to $-28x^2 + 96xy - 108x - 144y + 81 = 0$ with

$$y = \frac{(2x-9)(14x-9)}{48(2x-3)}$$

clearly with no integer solutions.

For $d = 3$ we get $96xy + 28y^2 - 192x - 256y + 256 = 0$ and

$$x = \frac{(7y-8)(y-8)}{24(y-2)}$$

with obvious solution $(0, 8)$. Less easier to find is $(3, -4)$. There are no other integral solutions.

For $d = 2$ the equation becomes $20x^2 + 96xy + 48y^2 - 252x - 336y + 441 = 0$. Solving for x we get

$$x = \frac{63}{10} - \frac{12}{5}y \pm \frac{1}{5}\sqrt{84y^2 - 336y + 441}$$

Hence $84y^2 - 336y + 441 = 84(y-2)^2 + 105$ must be square. Let $Y = y - 2$, then we have to solve Pell's equation $X^2 - 84Y^2 = 105$. ndexPell equation This equation has an infinity of solutions based on the fundamental solution $(21, 2)$, but a corresponding x is not integer. As we can see from

$$x = \frac{63}{10} - \frac{12}{5}(Y+2) \pm \frac{1}{5}X = \frac{15 - 24Y \pm 2X}{10}$$

For $d = 1$ we have the equation $32x^2 + 96xy + 60y^2 - 288x - 384y + 576 = 0$. Solving for x we get:

$$x = \frac{9}{2} - \frac{3}{2}y \pm \frac{1}{4}\sqrt{6y^2 - 24y + 36}$$

We want $6y^2 - 24y + 36 = 6(y-2)^2 + 12$ to be square. Let $Y = y - 2$, then we solve the Pell equation $X^2 - 6Y^2 = 12$. $(X, Y) = (6, 2)$ is a solution. The equation $X^2 - 6Y^2 = 1$ has fundamental solution $(5, 2)$.

We use the following result: If p, q is a solution of $x^2 - Dy^2 = N$, and r, s is a solution to $x^2 - Dy^2 = 1$, then $x = pr + qsD$, $y = ps + qr$ is also a solution of $x^2 - Dy^2 = N$, because $(pr + qsD)^2 - D(ps + qr)^2 = (p^2 - Dq^2)(r^2 - Ds^2)$.

We define the matrix

$$A = \begin{pmatrix} r & sD \\ s & r \end{pmatrix} = \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix}$$

Now we define

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = A^n \begin{pmatrix} p \\ q \end{pmatrix} = A^n \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

For each $n \geq 0$ we define $X = \pm X_n$ and $Y = \pm Y_n$. So for each $n \geq 0$ we find four solutions (x, y) with

$$y = Y + 2 \quad \text{and} \quad x = \frac{X - 6Y + 6}{4}$$

Results of a SAGE [1] program for calculating (x, y):

$(0, 4)$	$(-3, 4)$	$(3, 0)$	$(6, 0)$
$(-18, 24)$	$(-45, 24)$	$(21, -20)$	$(48, -20)$
$(-192, 220)$	$(-459, 220)$	$(195, -216)$	$(462, -216)$
$(-1914, 2160)$	$(-4557, 2160)$	$(1917, -2156)$	$(4560, -2156)$
$(-18960, 21364)$	$(-45123, 21364)$	$(18963, -21360)$	$(45126, -21360)$
$(-187698, 211464)$	$(-446685, 211464)$	$(187701, -211460)$	$(446688, -211460)$
$(-1858032, 2093260)$	$(-4421739, 2093260)$	$(1858035, -2093256)$	$(4421742, -2093256)$

Reference

Y [1] William Stein, David Joyner, SAGE : System for Algebra and Geometry Experimentation, Comm. Computer Algebra 39 (2005) 61-64.

See also: https://www.nieuwarchief.nl/serie5/pdf/naw5-2006-07-3-219.pdf

SAGE is now called SageMath, see https://www.sagemath.org/

Figure 19.1: QR-code link

Figure 19.2: www.sagemath.org

Remark

Those were the early days of SAGE, now SageMath. I was an early adapter of this alternative for the three M's: Maple, Mathematica and Magma.

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Problem B NAW 5/7 nr. 2 , December 2005

The problem

Introduction

Imagine a flea circus consisting of n boxes in a row, numbered $1, 2, ..., n$. In each of the first m boxes there is one flea ($m \leq n$). Each flea can jump upwards or forwards to boxes with a maximal distance $d = n - m$. For all fleas all $d + 1$ jumps have the same probability.

The director of the circus has marked m boxes to be special targets. On his sign all m fleas jump simultaneously.

1. Calculate the probability that after the jump exactly m boxes are occupied.

2. Calculate the probability that all the m marked boxes are occupied.

Solution

Part 1

Let $d = n - m$. The jumps of the fleas corresponds to a bipartite graph G. We can associate a (0,1)-matrix B of size m by n with this graph. We have $b_{ij} = 1$ if and only if $i \leq j \leq i + d$. A matching M with cardinality t corresponds in the matrix B to a set of t ones with no two of the ones on the same line. The total number of jumps with exactly m boxes occupied is the number of matchings with $|M| = m$ is $per(B)$, the permanent of B. See [1], p. 44.

The asked probability is $\frac{per(B)}{(d+1)^m}$.

Part 2

Let A be the set of marked boxes, so $A = \{a_1, a_2, ..., a_m\}$ is a subset of $\{1, 2, 3, ..., n\}$, with $1 \leq a_1 < a_2 < ... < a_m \leq n$ and ($m > 0$, $m \leq n$). A succesful jump of the fleas can be associated with a permutation of the elements of A. We are looking for permutations π of the elements of A with restrictions on permitted positions such that $k \leq \pi(k) \leq k + d$ for all $1 \leq k \leq m$. With this restrictions we can associate a (0,1)-matrix $C = [c_{ij}]$, where $c_{ij} = 1$, if and only if a_j is permitted in position i, meaning $i \leq a_j \leq i + d$.

Compare Problem 29 from NAW 5/3 nr. 1 March 2002.

We define S_C as the set of all permitted permutations, to be more precise

$$S_C = \{\pi | \prod_{i=1}^{m} c_{i\pi(i)} = 1\} \tag{20.1}$$

The number of elements of S_C can be calculated by summing over all possible π

$$|S_C| = \sum_{\pi} \prod_{i=1}^{m} c_{i\pi(i)} = per(C) \tag{20.2}$$

where $per(C)$ is the permanent of C. See [2].

So the asked probability is $\frac{per(C)}{(d+1)^m}$.

Figure 20.1: Dancing School Problems NAW Dec 2006

References

[1] Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, 1991.

21

Problem C NAW 5/7 nr. 3, September 2006

The problem

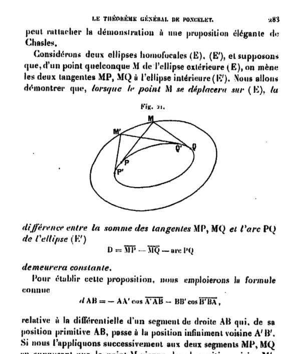
Introduction

Consider the triangle ABC inscribed in an ellipse. For given A the other vertices can be adjusted to maximize the circumference. Prove or disprove that this maximum circumference is independent of the position of A on the ellipse.

Solution

We show that the maximal circumference is independent of point A.

Let E and E' be ellipses with the same foci F_1 and F_2. Ellipse E' is inside E and sufficiently close to E. Let the points A, B, C and X be on E, such that the line segments AB, BC and CX are tangent to E'. In general the points A and X do not coincide, but if we shrink ellipse E' continously this will be eventually the case for $E' = E_0$. In this situation we have a triangle ABC inscribed in E with maximal circumference.



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peut rattacher la démonstration à une proposition élégante de Chasles.

Considérons deux ellipses homofocales (E), (E'), et supposons que, d'un point quelconque M de l'ellipse extérieure (E), on mène les deux tangentes MP, MQ à l'ellipse intérieure (E'). Nous allons démontrer que, *lorsque le point* M *se déplacera sur* (E), *la*

Fig. 21.

différence entre la somme des tangentes MP, MQ *et l'arc* PQ *de l'ellipse* (E')

$$D = \overline{MP} - \overline{MQ} - \text{arc } PQ$$

demeurera constante.

Pour établir cette proposition, nous emploierons la formule connue

$$d\,AB = -AA' \cos \overline{A'AB} - BB' \cos \overline{B'BA},$$

relative à la différentielle d'un segment de droite AB qui, de sa position primitive AB, passe à la position infiniment voisine A'B'. Si nous l'appliquons successivement aux deux segments MP, MQ en supposant que le point M vienne dans la position voisine M',

Figure 21.1: Illustration from note[1]

If we now move point A along ellipse E, we see that according to a theorem of Poncelet we always have a triangle inscribed in E and circumscribed around E_0 with by construction a maximal circumference. Chasles, Darboux and others proved that all this triangles have the same maximal circumference. See for example [1] Livre III, Chapitre III, part 176, p. 283. We follow the historical proof of Darboux, using 'infinitesimal' arguments. Let triangle ABC be defined as above. We have tangents AP and AQ with P and Q on ellipse E_0. Triangle ABC is a 'billiard triangle', meaning the tangents make equal angles tot the normal of E in point A. We move A to A' over an 'infinitesimal' distance and the corresponding points P and Q move to P' and Q'.

Figure 21.2: NAW 2006-3

Taking in account the properties of tangents we have

$$d(AP) = -AA'cos(A'AP) + PP'$$

and

$$d(AQ) = -AA'cos(A'AQ) - QQ'$$

and with the fact that the angles $A'AP$ and $A'AQ$ are supplementary, we get

$$d(AP + AQ) = PP' - QQ' = d(arc(PQ))$$

Hence the difference $D = (AP + AQ) - arc(PQ)$ is constant.

Doing this for all vertices of triangle ABC this leads to $3D = O - O'$, O being the circumference of ABC and O' the perimeter of the ellipse E_0.

Conclusion: The maximal circumference is independent of the position of A.

This problem can easily be generalized to an n-sided (convex) polygon inscribed in an ellipse for $n \geq 3$.

See: https://www.nieuwarchief.nl/serie5/pdf/naw5-2007-08-3-232.pdf

Figure 21.3: NAW September 2007

Remark

For a treatment independent of Poncelet's Theorem see George Lion, Variational Aspects of Poncelet's Theorem, Geometricae Dedicata 52, 105-118, 1994.

References

[1] Darboux, G: Principes de Géométrie analytique, Gauthier-Villars, Paris, 1917. Available in facsimile: https://gallica.bnf.fr

22

Problem C NAW 5/7 nr. 4, December 2006

The problem

Introduction

Let G be a finite group of order $p+1$ with p a prime. Show that p divides the order of $Aut(G)$ if and only if p is a Mersenne prime, that is, of the form $2^n - 1$, and G is isomorphic to $(Z/2)^n$.

Solution

Let p be a Mersenne prime with $p = 2^n - 1$ and G be isomorphic to $(Z_2)^n$, so G is an elementary Abelian group of order 2^n. It is a well known fact that the group of automorphisms of the elementary Abelian group of order q^r is of order $(q^r - 1)(q^r - q)...(q^r - q^{r-1})$, the order of $GL(r, q)$. Hence $p = 2^n - 1$ is a divisor of $|Aut(G)|$. Let now p be a divisor of $|Aut(G)|$. $|G| = p + 1$, so there are p elements of G not equal the identity e, say $g_1, g_2, ..., g_p$. Clearly $p > 2$, so $p+1$ is even, so according to the first Sylow Theorem there is a subgroup of G of order 2, and hence there is an element g of order 2. As p is a divisor of the order of the automorphism group of G, we need all possible automorphisms with $g \to g_i$, $i = 1, 2, ..., p$, hence all elements g_i are of order 2.

See for instance Marshall Hall, Jr. The Theory of Groups Chapter 4.

So G is isomorphic to $(Z/2)^n$ with $p + 1 = 2^n$ and hence $p = 2^n - 1$ is a Mersenne prime.

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Problem A NAW 5/8 nr. 1, April 2007

The problem

Introduction

Define the sequence $\{u_n\}$ by $u_1 = 1$, $u_{n+1} = 1 + (n/u_n)$. Prove or disprove that

$$u_n - 1 < \sqrt{n} \leq u_n$$

Solution

By definition we have $u_1 = 1$, $u_{n+1} = 1 + (n/u_n)$ and hence $u_n(u_{n+1} - 1) = n$.

So $\sqrt{n} = \sqrt{u_n(u_{n+1} - 1)}$ is the geometric mean of u_n and $u_{n+1} - 1$ and therefor

$$u_{n+1} - 1 \leq \sqrt{n} \leq u_n$$

for all $n \geq 1$.

Further we have for $n \geq 2$

$$u_n - 1 \leq \sqrt{n-1} < \sqrt{n}$$

This inequality holds also for $n = 1$, so we proved

$$u_n - 1 < \sqrt{n} \leq u_n$$

for all $n \geq 1$.

24

Problem B NAW 5/8 nr. 1, April 2007

The problem

Introduction

Given a non-degenerate tetrahedron (whose vertices do not all lie in the same plane), which conditions have to be satisfied in order that the altitudes intersect at one point?

Solution

Let T be such a tetrahedron with vertices A_0, A_1, A_2 and A_3 in a Euclidean space E. We define $\bar{a}_i = \vec{OA_i}$ and

$$\bar{e}_{ij} = \bar{a}_i - \bar{a}_j \tag{24.1}$$

for $i \neq j$. The vector $\bar{e}_{ij}$ is a direction vector of the edge A_iA_j.

The altitude h_i passing through A_i is determined by the following equations

$$\begin{aligned} \bar{e}_{jk} \cdot (\bar{a}_i - \bar{x}) &= 0 \\ \bar{e}_{jl} \cdot (\bar{a}_i - \bar{x}) &= 0 \\ \bar{e}_{kl} \cdot (\bar{a}_i - \bar{x}) &= 0 \end{aligned} \tag{24.2}$$

with $\{i, j, k, l\} = \{0, 1, 2, 3\}$.

Note that we need only two of them to determine h_i.

We now proof the following Lemma:

An altitude h_i intersects with an altitude h_j if and only if

$$\bar{e}_{kl} \cdot \bar{e}_{ij} = 0 \tag{24.3}$$

Proof: From (2), the altitude h_i is determined by the equations

$$\begin{aligned} \bar{e}_{jk} \cdot (\bar{a}_i - \bar{x}) &= 0 \\ \bar{e}_{kl} \cdot (\bar{a}_i - \bar{x}) &= 0 \end{aligned}$$

and the altitude h_j is determined by the equations

$$\begin{aligned} \bar{e}_{li} \cdot (\bar{a}_j - \bar{x}) &= 0 \\ \bar{e}_{kl} \cdot (\bar{a}_j - \bar{x}) &= 0 \end{aligned}$$

Let P be on h_i and h_j. Let $\bar{p}$ be the point vector of P. Then $\bar{e}_{kl} \cdot (\bar{a}_i - \bar{p}) = 0$ and $\bar{e}_{kl} \cdot (\bar{a}_j - \bar{p}) = 0$ and hence

$$\bar{e}_{kl} \cdot (\bar{a}_i - \bar{a}_j) = \bar{e}_{kl} \cdot \bar{e}_{ij} = 0$$

From (3) it follows that $\bar{e}_{kl} \cdot \bar{a}_i = \bar{e}_{kl} \cdot \bar{a}_j$ and so two of the four equations are equal. Three planes intersect in one point unless they are parallel to a line. This is clearly not the case since T is non-degenerate and the vectors $\bar{e}_{jk}$, $\bar{e}_{kl}$ and $\bar{e}_{il}$ are independent. So h_i and h_j must have a point in common.

For reasons of symmetry the same holds for the altitudes h_k and h_l.

Definition: A tetrahedron is called orthocentric if the altitudes intersect in one point.

Theorem: The following statements are equivalent:

$i)$ T is orthocentric.

$ii)$ All opposite edges are orthogonal.

Proof: $i) \Rightarrow ii)$. This follows immediately from the lemma.

$ii) \Rightarrow i)$. We now have

$$\bar{e}_{ij} \cdot \bar{e}_{kl} = \bar{e}_{ik} \cdot \bar{e}_{jl} = \bar{e}_{il} \cdot \bar{e}_{jk} = 0$$

So by the lemma, any two altitudes intersect. The four altitudes are not in the same plane, so there must be a common point.

25

Problem A NAW 5/8 nr. 2, August 2007

The problem

Introduction

1. Find the largest number c such that all natural numbers n satisfy

$$n\sqrt{2} - \lfloor n\sqrt{2} \rfloor \geq \frac{c}{n}$$

2. For this c, find all natural numbers n such that

$$n\sqrt{2} - \lfloor n\sqrt{2} \rfloor = \frac{c}{n}$$

Solution

Let $p = \lfloor n\sqrt{2} \rfloor$ and $q = n$, so we have to find the largest c for which

$$\sqrt{2} - \frac{p}{q} \geq \frac{c}{q^2} \qquad (25.1)$$

holds for all natural numbers q.

We define a function $f : x \to x^2 - 2$. The equation $f(x) = 0$ has a solution $x = \sqrt{2}$. We note that $\frac{p}{q}$ is an approximation of $\sqrt{2}$ and that $1 \leq \frac{p}{q} < \sqrt{2}$.

Let M be the maximal value of $f'(x) = 2x$ in the interval $[1, \sqrt{2}]$, so $M = 2\sqrt{2}$. Now $f(\frac{p}{q}) = (\frac{p}{q})^2 - 2 = \frac{p^2 - 2q^2}{q^2}$, hence $\left| f(\frac{p}{q}) - f(\sqrt{2}) \right| \leq \frac{1}{q^2}$.

By the mean-value theorem we get:

$$f(\frac{p}{q}) - f(\sqrt{2}) = f'(\xi)(\frac{p}{q} - \sqrt{2})$$

for some ξ in the interval $[1, \sqrt{2}]$.

Hence

$$\sqrt{2} - \frac{p}{q} = \left|\frac{p}{q} - \sqrt{2}\right| \geq \frac{1}{Mq^2} = \frac{\frac{1}{4}\sqrt{2}}{q^2}$$

Mutatis mutandi we have found $c = \frac{1}{4}\sqrt{2}$.

For this c there clearly is no n satisfying the equality of question 2.

Remark

According to [Hardy][1] the numbers $\sqrt{5}$ and $2\sqrt{2}$ play a crucial role in approximations of irrational numbers by rationals. For instance the Theorem: Any irrational $\xi \neq \frac{1}{2}(\sqrt{5} - 1)$ has an infinity of rational approximations for which

$$\left|\frac{p}{q} - \xi\right| < \frac{1}{2q^2\sqrt{2}} = \frac{\frac{1}{4}\sqrt{2}}{q^2}$$

Interesting, isn't it?

[1] G.H. Hardy and E.M. Wright. *An introduction to the theory of numbers*. Clarendon Press Oxford University Press, Oxford New York, 1979. ISBN 0198531710

Reference

[Hardy] Hardy, Wright, An Introduction to the Theory of Numbers, 5th edition, Oxford.

26

Problem C NAW 5/8 nr. 4, December 2007

The problem

Introduction

Let G be a finite group with n elements. Let c be the number of pairs $(g_1, g_2) \in G \times G$ such that $g_1 g_2 = g_2 g_1$. Show that either G is commutative or that $8c \leq 5n^2$. Show that if $8c = 5n^2$ then 8 divides n.

Solution

We need some elementary group theory and notation. Let $Z(g)$ be the centralizer of $g \in G$ and $K(g)$ the conjugacy class containing g. Z is the center of G.

We now have $|G| = n$, $c = \sum_{g \in G} |Z(g)|$ and

$$|K(g)| = [G : Z(g)] = \frac{|G|}{|Z(g)|} = \frac{n}{|Z(g)|}$$

We write the ratio

$$\begin{aligned} r &= \frac{c}{n^2} = \frac{\sum_{g \in G} |Z(g)|}{n^2} = \\ &= \frac{1}{n^2} \cdot \sum_{g \in G} \frac{n}{|K(g)|} = \\ &= \frac{1}{n} \cdot \sum_{g \in G} \frac{1}{|K(g)|} = \frac{k}{n} \end{aligned}$$

where k is the number of conjugacy classes.

We note that $r = 1$ if and only if G is commutative. So from now on let G be a non Abelian group.

We proof the following lemma:

The order of G/Z can not be a prime number.

Proof: As groups with order a prime are cyclic it is enough to proof that G/Z can not be cyclic. Suppose G/Z be cyclic generated by Zx. We get

$$G = Z \cup Zx \cup (Zx)^2 \cup (Zx)^3 \cup \cdots = Z \cup Zx \cup Zx^2 \cup Zx^3 \cup \cdots$$

and now arbitrary elements $g_1 = z_1 x^i$ and $g_2 = z_2 x^j$ clearly commute. This is a contradiction.

In order to maximise the number of conjugacy classes k we must maximise $|Z|$ the number of conjugacy classes with only one element. From the lemma it follows that $|G/Z| \geq 4$ and so $|Z| \leq \frac{1}{4}|G|$. The other conjugacy classes must have 2 or more elements. Hence

$$k \leq \frac{1}{4}|G| + \frac{1}{2} \cdot \frac{3}{4}|G| = \frac{5}{8}n$$

so that

$$r \leq \frac{5}{8}$$

and therefor

$$8c \leq 5n^2$$

.

If $r = \frac{5}{8}$ and hence $k = \frac{5}{8}n$ it is trivial that 8 is a divisor of n.

Figure 26.1: NAW 2007-4 solution

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Problem A NAW 5/9 nr. 1, March 2008

The problem

Introduction

Denote the fractional part of a positive real number x by $\{x\}$.
Evaluate the following double integral:

$$\int_0^1 \int_0^1 \left\{\frac{x}{y}\right\} \left\{\frac{y}{x}\right\} dxdy$$

Solution

Let $z = \left\{\frac{x}{y}\right\} \left\{\frac{y}{x}\right\}$, then

$$\begin{aligned} z &= \left(\frac{x}{y} - \left\lfloor\frac{x}{y}\right\rfloor\right)\left(\frac{y}{x} - \left\lfloor\frac{y}{x}\right\rfloor\right) = 1 - \frac{x}{y}\left\lfloor\frac{y}{x}\right\rfloor - \frac{y}{x}\left\lfloor\frac{x}{y}\right\rfloor + \left\lfloor\frac{x}{y}\right\rfloor\left\lfloor\frac{y}{x}\right\rfloor \\ &= \begin{cases} 0 & \text{if} \quad x = y \\ 1 - \frac{x}{y}\lfloor\frac{y}{x}\rfloor & \text{if} \quad x < y \\ 1 - \frac{y}{x}\lfloor\frac{x}{y}\rfloor & \text{if} \quad x > y \end{cases} \end{aligned}$$

for $0 < x \leq 1$ and $0 < y \leq 1$.

By symmetry we have

$$I = \int_0^1 \int_0^1 z\,dxdy = 2\int_0^1 \int_0^y z\,dxdy = 2\int_0^1 \int_0^y (1 - \left\lfloor\frac{y}{x}\right\rfloor \frac{x}{y})dxdy$$

Now if $n \leq \frac{y}{x} < n+1$ we have $\lfloor\frac{y}{x}\rfloor = n$, $z = 1 - n\frac{x}{y}$ and $\frac{1}{n+1}y < x \leq \frac{1}{n}y$.

We define

$$I_n = \int_0^1 \int_{\frac{1}{n+1}y}^{\frac{1}{n}y} (1 - n\frac{x}{y}) dxdy$$

We can easily check that

$$I_n = \frac{1}{4}(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2}) = \frac{1}{4n(n+1)^2}$$

and

$$I = 2\sum_{n=1}^{\infty} I_n = 2\sum_{n=1}^{\infty} \frac{1}{4n(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{2n(n+1)^2} = 1 - \frac{\pi^2}{12}$$

28

Problem B NAW 5/10 nr.2, June 2009

The problem

Introduction

A magic $n \times n$ matrix of order r is an $n \times n$ matrix whose entries are non-negative integers and whose row and column sums are all equal to r. Let $r > 0$ be an integer. Show that a magic $n \times n$ matrix of order r is the sum of r magic $n \times n$ matrices of order 1.

Solution

Let A be an $n \times n$ magic matrix of order r. We want to assert that A has n positive entries with no two positive entries on a line. This looks trivial but it is not. We need a famous Minimax Theorem of König. This theorem is known in various equivalent forms and there are all kinds of proof in literature. We follow the notation of [1][1].

Let A be an $m \times n$ matrix with elements from a ring R. The minimal number of lines in A that cover all the non-zero elements of A is equal the the maximal number of non-zero elements in A with no two of the non-zero elements on a line.

We now return to our magic matrix A. If A does not have n positive entries with no two on a line, then by König's theorem we could cover all the positive entries in A with x rows and y columns, where $x + y < n$. All line sums are equal to r, so $(x + y) \cdot r$ counts up to at least nr. We now have $n \leq x + y < n$, which is a contradiction.

[1] Richard Brualdi and Herbert Ryser. *Combinatorial matrix theory*. Cambridge University Press, Cambridge England New York, 1991. ISBN 0521322650

Let P_1 be the permutation matrix of order n with ones in the same position as those occupied by the n positive entries of A. Let c_1 be the smallest one of those positive entries. Then clearly $X_1 = A - c_1 P_1$ is a magic matrix of order $r - c_1$ with at least one more zero and $A = c_1 P_1 + X_1$.

Applying the same argument on X_1 gives $X_2 = X_1 - c_2 P_2$, iterating until we get an all zero matrix X_t, we obtain

$$A = c_1 P_1 + c_2 P_2 + \cdots + c_t P_t$$

Multiplying by J, the $n \times n$ matrix with all ones, we get

$$c_1 + c_2 + \cdots + c_t = r$$

We now note that $c_i P_i$ equals the sum of c_i copies of P_i and P_i is a magic matrix of order 1. So we have decomposed A in r magic matrices of order 1.

References

[1] R.A Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, 1991.

29

Problem B NAW 5/10 nr.3, September 2009

The problem

Introduction

Find all functions $f : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ such that

$$f(x+y) \geq f(x) + yf(f(x)) \tag{29.1}$$

for all x and y in $\mathbf{R}_{>0}$

Solution

Let us rewrite (1) a bit to:

$$f(x+y) = f(x) + yf(f(x)) + \epsilon(x,y) \tag{29.2}$$

with $\epsilon(x,y) \geq 0$ and $\{lim_{y\downarrow 0}\epsilon(x,y) = 0$.

This reminds us to the lecture notes of Prof. Van der Blij in the early sixties:

$$f(x+h) = f(x) + hf'(x) + \epsilon^*(x,h) \tag{29.3}$$

with $\lim_{h\to 0} \frac{\epsilon^*(x,h)}{h} = 0$.

We easily see that a solution of the functional-differential equation

$$f'(x) = f(f(x)) \tag{29.4}$$

with domain $D_f \subset \mathbf{R}_{>0}$ can satisfy (2).

According to the references [1] and [2] solutions with $x > 0$ and $f(x) > 0$ exist, but not with $D_f = \mathbf{R}_{>0}$. A typical solution has the following properties:

1. We have an $x_0 > 0$ with $f(x_0) < x_0$
2. There is a σ depending on x_0 with $0 < \sigma \leq 1$ and $f(\sigma) = \sigma$.
3. We have a $b > \sigma$ with $f(b) < b$
4. There is a $c > b$ with $f(c) = c$.
5. The function f is increasing.
6. $D_f =< 0, c] \subset \mathbf{R}_{>0}$.

Those solutions can not be continued for values of x larger than c

Our conclusion is that there are no functions f with domain $D_f = \mathbf{R}_{>0}$ satisfying (1).

For an elegant solution see: QR-code.

Figure 29.1: QR-code link

References

[1] Eder, Elmar: The functional differential equation $x'(t) = x(x(t))$. J. Diff. Eq. 54 (1984), 390-400.

[2] Wang Ke: On the Equation $x'(t) = f(x(x(t)))$. Funcialaj Ekvacioj, 33 (1990), 405-425.

Part II

Special Problems

30
NAW Problem 26

Abstract

Does there exist a triangle with sides of integral lengths such that its area is equal to the square of the length of one of its sides?
The answer is no.

Figure 30.1: NAW December 2001

The problem

Introduction.

Does there exist a triangle with sides of integral lengths such that its area is equal to the square of the length of one of its sides?

Solution 1.

We can scale to integral sides easily, so suppose we have a triangle with rational sides a, b and c with area A. Then the famous Heron formula gives

$$\begin{aligned} (4A)^2 &= (a+b+c)(a+b-c)(a-b+c)(-a+b+c) \\ &= (x+c)(x-c)(y+c)(-y+c) \\ &= (x^2-c^2)(-y^2+c^2) \end{aligned}$$

with $x = a + b$ and $y = a - b$.

Without loss of generality we may state that $A = 1$ and $c = 1$. So we have

$$(x^2 - 1)(y^2 - 1) = -16$$

Does this equation have rational solutions?

We make substitutions $X = x$ and $Y = y(x^2 - 1) - x^2$. Rearanging we get

$$(Y + X^2)^2 - (X^2 - 1)^2 = -16(X^2 - 1)$$

Substitution of $X = 1/U - Y$ and solving for U means solving a quartic with discriminant $-2Y^3 - 18Y^2 + 34Y + 306$. So we are looking for a rational solution of

$$Z^2 = -2Y^3 - 18Y^2 + 34Y + 306$$

$Y = -1$ and $Z = 16$ represents a solution, so we can write

$$v^2 = 2u^3 - 12u^2 - 64u + 256$$

where $v = Z$ and $u = -Y - 1$.

This equation of an elliptic curve can be transformed into a Weierstrass equation (N.B.: new meaning of x and y):

$$y^2 - 4xy + 64y = x^3 - 16x^2$$

Which can be reduced to minimal form

$$y_1^2 + x_1y_1 + y_1 = x_1^3 - x_1^2 - x_1$$

This means that we have the elliptic curve (17 A 4 [1,-1,1,-1,0] 0 4) from the appropriate Cremona table.

https://johncremona.github.io/ecdata

This curve has rank zero, so in the torsion group we find all rational solutions: (0,0), (0,-1) and (1,-1). It is easily verified that this result gives no solutions to our original problem.

Figure 30.2: https://johncremona.github.io/ecdata

Solution 2.

Suppose a triangle ABC exists with integral sides a, b and c with basis c, area c^2 and height $CD = 2c$. Let $BD = d$, then

$d^2 = a^2 - 4c^2$ and $d = \sqrt{a^2 - 4c^2}$. We consider the case that ABC is obtuse (The acute case is left as an excercise).

$$\begin{aligned} b^2 &= (c+d)^2 + 4c^2 \\ &= (c + \sqrt{a^2 - 4c^2})^2 + 4c^2 \\ &= a^2 + c^2 + 2c\sqrt{a^2 - 4c^2} \end{aligned}$$

Here a, b and c are integral, so also $d = \sqrt{a^2 - 4c^2}$ must be an integer and therefore the triangles BDC and ADC are Pythagorean.

A well known result states that a Pythagorean triangle can be parametrized. We leave out some of the details. For BDC we have $a = BC = u^2 + v^2$, $BD = u^2 - v^2$ and $CD = 2uv$, with integers u, v and $u > v$. In triangle ADC we have $AD = s^2 - t^2$ and $CD = 2st$, with integers s, t and $s > t$. While $AB = c = uv$ we have

$$\begin{aligned} s^2 - t^2 &= u^2 - v^2 + uv \\ st &= uv \end{aligned}$$

Dividing the lefthand side of the first equation by st and the righthand side by uv we get

$$\frac{s}{t} - \frac{t}{s} = \frac{u}{v} - \frac{v}{u} + 1$$

Substitution of $y = \frac{s}{t}$ and $x = \frac{u}{v}$ while $st = uv$ gives

$$y - \frac{1}{y} = x - \frac{1}{x} + 1$$

By multiplying with xy we get

$$x^2y - xy^2 + xy + x - y = 0$$

This cubic can eventually be transformed into a Weierstrass equation of an elliptic curve by Nagell's algorithm. At first we tried this by hand, but the famous Apecs lib for Maple V from Ian Connell did it within seconds. The command $Gcub(0, 1, -1, 0, 0, 1, 0, 1, -1, 0, 0, 0)$; returned among others:

present curve is, $A17 = [1, -1, 1, -1, 0]$

Meaning that we have the same elliptic curve as in solution 1.

Connell, Apecs (arithmetic of plane elliptic curves), a program written in Maple, available via anonymous ftp from math.mcgill.ca in /pub/apecs (1997). This information is obsolete. You eventually can find the Apecs file on the Internet Archive Wayback Machine.

Solution 3.

We can scale to integral sides easily, so suppose we have a triangle with rational sides a, b and c with area A. Then the Heron formula gives

$$\begin{aligned}(4A)^2 &= (a+b+c)(a+b-c)(a-b+c)(-a+b+c)\\ &= (x+c)(x-c)(y+c)(-y+c)\\ &= (x^2-c^2)(-y^2+c^2)\end{aligned}$$

with $x = a+b$ and $y = a-b$.

Without loss of generality we may state that $A = 1$ and $c = 1$. So we have

$$(x^2-1)(y^2-1) = -16$$

Does this equation have rational solutions?

Making the substitutions $U = x$ and $V = y(x^2-1) - x^2$ and rearanging we get

$$(V+U^2)^2 - (U^2-1)^2 = -16(U^2-1)$$

and so

$$2U^2V + 18U^2 + V^2 - 17 = 0$$

with solution $U = 1$ and $V = -1$.

The Apecs command $Gcub(0,2,0,0,18,0,1,0,0,-17,1,-1)$; returned among other information:

$$\text{present curve is, } A17 = [1,-1,1,-1,0]$$

I'm still able to run Apecs6 for Maple V in Windows XP running in VirtualBox, even with Maple 9. Modern software has taken over. See for instance SageMath.

Meaning that we have a well known elliptic curve of rank zero while the order of the torsion group equals 4. So the members of the torsion group (1,-1), (0,0) and (0,-1) are the only rational solutions. It is easily verified that this result gives no solutions to our original problem.

Solution 4.

A Heron triangle is a triangle with sides of integral length and integral area. According to K.R.S Sastry [1] every Heron triangle with sides a, b and c can be parametrized as follows:

Let λ be a rational number such that $0 < \lambda \leq 2$.

$$(a,b,c) = (2(m^2+\lambda^2 n^2), (2+\lambda)(m^2-2\lambda n^2), \lambda(m^2+4n^2)),$$

m, n being relative prime natural numbers such that $m > \sqrt{2\lambda}.n$. For the sides we have $a \geq c$ and $gcd(a,b,c) = 1$. The area of this triangle $\Delta = 2\lambda(2+\lambda)mn(m^2-2\lambda n^2)$.

As one can see Δ is always a multiple of b. So looking for a solution of our problem we have to consider $\Delta = b^2$, so

$$2\lambda mn = (2+\lambda)(m^2-2\lambda n^2)$$

Dividing this equation by n^2 gives

$$2\lambda\frac{m}{n} = (2+\lambda)((\frac{m}{n})^2 - 2\lambda)$$

Making the substitutions $U = \lambda$ and $V = \frac{m}{n}$ we get

$$UV^2 - 2U^2 - 2UV + 2V^2 - 4U = 0$$

with obvious solution $U = 0$ and $V = 0$.

This cubic can eventually be transformed into a Weierstrass equation of an elliptic curve by Nagell's algorithm.

The Apecs command $Gcub(0,0,1,0,-2,-2,2,-4,0,0)$; returned among other information:

$$\text{present curve is, } A17 = [1,-1,1,-1,0]$$

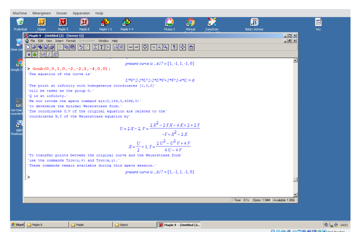


Figure 30.3: Apecs

Meaning that we have the same well known elliptic curve of rank zero while the order of the torsion group equals 4. So the members of the torsion group (1,-1), (0,0) and (0,-1) are the only rational solutions. It is easily verified that this result gives no solutions to our original problem.

[1] K.R.S Sastry, Heron Triangles: A Gergonne-Cevian-and-Median Perspective, Forum Geometricorum Volume 1 (2001) 17-24

Solution 5.

We can scale to integral sides easily, so suppose we have a triangle ABC with rational sides a, b and c with rational area Δ and

$\angle ACB = \theta$. Since the area $\Delta = \frac{1}{2}ab\sin\theta$ is rational, $\sin\theta$ must be a rational number.

According to the law of cosines, $c^2 = a^2 + b^2 - 2ab\cos\theta$, so also $\cos\theta$ must be rational. Rational points on the unit circle can be parametrized as follows:

$$(\cos\theta, \sin\theta) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$$

where the point (-1,0) or $\theta = \pi$ is excluded.

Without loss of generality we may state that $\Delta = 1$ and $c = 1$. So we have

$$\begin{aligned} 1 &= \frac{1}{2}ab\sin\theta \\ 1 &= a^2 + b^2 - 2ab\cos\theta \end{aligned}$$

and therefore $ab = \frac{2}{\sin\theta}$. This results in

$$a^2 + b^2 = 1 + \frac{2(1-t^2)}{t}$$

Let's try to investigate on this last expression. In triangle ABC we have height $CD = 2$ and let $AD = x$. We can treat the obtuse and acute case in one formula (the other case is trivial)

$$\begin{aligned} a^2 + b^2 &= (x-1)^2 + 4 + x^2 + 4 \\ &= (1-x)^2 + 4 + x^2 + 4 \\ &= 2x^2 - 2x + 9 \end{aligned}$$

so $2x^2t - 2xt + 9t = -2t^2 + t + 2$

It is clear that rational a and b give a rational x, but the converse is apparantly not true. See for instance $x = \frac{1}{2}$, meaning $a = b = \frac{1}{2}\sqrt{17}$.

Making the substitutions $U = x$ and $V = t$ and rearanging we get

$$2U^2V - 2UV + 2V^2 + 8V - 2 = 0$$

This cubic can be transformed into the minimal form of a Weierstrass equation of an elliptic curve by Nagell's algorithm.

$$Y^2 + YX + Y = X^3 - X^2 - 6X - 4$$

Meaning that we have the elliptic curve $(17A2[1, -1, 1, -6, -4], 0, 4)$ of rank zero while the order of the torsion group equals 4 from the Cremona table. So the members of the torsion group (3,-2), (-1,0) and (-5/4,1/8) are the only rational solutions. It is easily verified that this result gives no solutions to our original problem ($x = \frac{1}{2}$ and ($t = \frac{1}{4}$ or $t = -4$)).

Solution 6.

We can scale to integral sides easily, so suppose we have a triangle ABC with rational sides a, b and c with rational area Δ. Without loss of generality we may state that $\Delta = 1$ and $c = 1$. Let the height $CD = 2$ and $AD = x$. So, even without a picture we can see that

$$\begin{aligned} a^2 &= (1-x)^2 + 4 = (x-1)^2 + 4 \\ b^2 &= x^2 + 4 \end{aligned}$$

In our first attempt we eliminated x from this equations. This resulted in

$$(a^2 - b^2)^2 - 2(a^2 + b^2) + 17 = 0$$

Trying to solve this elegant equation we have to introduce a variable say x with $x^2 = b^2 - 4$. So we better take the shortcut.

As easily can be seen the rational solutions of our second equation can be parametrized with rational t by

$$x = \frac{4t}{1-t^2}, \quad b = \frac{2(t^2+1)}{1-t^2}$$

Substitution of x in the first equation gives

$$a^2 = \frac{5t^4 + 8t^3 + 6t^2 - 8t + 5}{(1-t^2)^2}$$

So we are looking for the rational solutions of the quartic

$$y^2 = 5t^4 + 8t^3 + 6t^2 - 8t + 5$$

with integer solution (1,4).

The command $Quar(5,8,6,-8,5,1,4)$ from the Apecs package for Maple V returned among other information:

$$\text{present curve is, } A17 = [1,-1,1,-1,0]$$

Meaning that we have again the well known elliptic curve from the Cremona table (17 A 4 [1,-1,1,-1,0] 0 4) of rank zero while the order of the torsion group equals 4. So the members of the torsion group (1,-1), (0,0) and (0,-1) are the only rational solutions. It is easily verified that this result gives no solutions to our original problem.

Solution 7.

We can scale to integral sides easily, so suppose we have a triangle ABC with rational sides a, b and c with rational area Δ. Without loss of generality we may state that $\Delta = 1$ and $c = 1$. Let the height $CD = 2$ and $BD = x$. As we have seen before the rectangular triangle BDC has a corresponding Pythagorean triangle which can be parametrized using the integrals u and v. So we have

$$x = \frac{u^2 - v^2}{uv}, \quad CD = \frac{2uv}{uv} = 2 \quad \text{and} \quad a = BC = \frac{u^2 + v^2}{uv}.$$

and

$$\begin{aligned} b^2 &= 5 + 2x + x^2 \\ &= 5 + 2\frac{u^2 - v^2}{uv} + (\frac{u^2 - v^2}{uv})^2 \\ &= \frac{u^4 + 2u^3v + 3u^2v^2 - 2uv^3 + v^4}{u^2v^2} \end{aligned}$$

Dividing by v^4 and substituting $U = \frac{u}{v}$, we end up by searching rational solutions of

$$V^2 = U^4 + 2U^3 + 3U^2 - 2U + 1$$

with obvious solution (0,1).

The command $Quar(1,2,3,-2,1,0,1)$ from the Apecs package for Maple V returned among other information:

$$\text{present curve is, } A17 = [1,-1,1,-1,0]$$

And we did it again! Meaning that we have once again the well known elliptic curve from the Cremona table (17 A 4 [1,-1,1,-1,0] 0 4) of rank zero while the order of the torsion group equals 4. So the members of the torsion group (1,-1), (0,0) and (0,-1) are the only rational solutions. It is easily verified that this result gives no solutions to our original problem.

Solution 8.

A Heron triangle is a triangle with sides of integral length and integral area. We use a parametric representation of the Heronian triangles as found in [1]

$$\begin{aligned} a &= n(m^2+k^2) \\ b &= m(n^2+k^2) \\ c &= (m+n)(mn-k^2) \\ \Delta &= kmn(m+n)(mn-k^2) \end{aligned}$$

For any integers m, n and k with $mn > k^2 > \frac{m^2 n}{(2m+n)}$, $\gcd(m,n,k) = 1$ and $m \geq n \geq 1$ we have one member of each simularity class of the Heronian triangles.

As one can see Δ is always a multiple of c. So looking for a solution of our problem we have to consider $\Delta = c^2$, so

$$kmn(m+n)(mn-k^2) = (m+n)^2(mn-k^2)^2$$

Or

$$kmn = (m+n)(mn-k^2)$$

Dividing this equation by n^3 gives

$$\frac{k}{n} \cdot \frac{m}{n} = (\frac{m}{n}+1)(\frac{m}{n} - (\frac{k}{n})^2)$$

Making the substitutions $U = \frac{m}{n}$ and $V = \frac{k}{n}$ we get

$$UV^2 - U^2 + UV + V^2 - U = 0$$

with obvious solution $U = 0$ and $V = 0$.

This cubic can eventually be transformed into a Weierstrass equation of an elliptic curve by Nagell's algorithm. The Apecs package for Maple V from Ian Connell did this with no pain. The command $Gcub(0,0,1,0,-1,1,1,-1,0,0)$; returned among other information:

$$\text{present curve is, } A17 = [1,-1,1,-1,0]$$

Meaning that we have once again the well known elliptic curve of rank zero while the order of the torsion group equals 4. So the members of the torsion group (1,-1), (0,0) and (0,-1) are the only rational solutions. It is easily verified that this result gives no solutions to our original problem.

[1] Buchholz, R.H., Perfect Pyramids, Bull. Austral. Math. Soc. 45, nr 3, 1992.

Conclusion.

There is no such triangle.

With thanks to Ian Connell, John Cremona, James Milne and Dave Rusin.

31

NAW Problem 29, January 2003

The Problem

Let n and h be natural numbers with $n > 0$ and A be a subset of $\{1, 2, ..., n + h\}$ with size n. Count the number of bijective maps $\pi : \{1, 2, ..., n\} \to A$ such that $k \leq \pi(k) \leq k + h$ for all $1 \leq k \leq n$.

Solution

Let $A = \{a_1, a_2, ..., a_n\}$ be a subset of $\{1, 2, 3, ..., n + h\}$, with $1 \leq a_1 < a_2 < ... < a_n \leq n + h$ and ($n > 0$, $h \geq 0$). We are looking for permutations π of the elements of A with restrictions on permitted positions such that $k \leq \pi(k) \leq k + h$ for all $1 \leq k \leq n$. With this restrictions we can associate a (0,1)-matrix $B = [b_{ij}]$, where $b_{ij} = 1$, if and only if a_j is permitted in position i, meaning $0 \leq a_j - i \leq h$.

We define S_B as the set of all permitted permutations, to be more precise

$$S_B = \{\pi | \prod_{i=1}^{n} b_{i\pi(i)} = 1\}$$

The number of elements of S_B can be calculated by

$$|S_B| = \sum_{\pi} \prod_{i=1}^{n} b_{i\pi(i)} = per(B)$$

where $per(B)$ is the permanent of B.

Example

Let $n = 4$, $h = 3$ and $A = \{2, 3, 5, 6\}$. We can easily see that in this case we have

$$B = \begin{pmatrix} 1\,1\,0\,0 \\ 1\,1\,1\,0 \\ 0\,1\,1\,1 \\ 0\,0\,1\,1 \end{pmatrix}$$

and $per(B) = 5$, so there are 5 permitted permutations. Being $(2, 3, 5, 6)$, $(3, 2, 5, 6)$, $(2, 3, 6, 5)$, $(3, 2, 6, 5)$ and $(2, 5, 3, 6)$.

Implementation

An implementation of the algorithm can be found on the website of the author:

https://www.jaapspies.nl/mathfiles/problem29.c

For a given n and h this program calculates for all possible subsets A the number of allowed bijective maps.

Literature

[1] R.A Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, 1991.

[2] H. Minc, Permanents, Reading, MA: Addison-Wesley, 1978[1].

[1] Henryk Minc. *Permanents*. Addison-Wesley, Reading, Mass., 1978

32

The Dancing School Problems, Februari 14, 2003

We give results related to problem 29 of the NAW. There are connections to Mathematical Recreation and Graph Theory.

The Problem

Introduction.

The Dancing School Poblem:

Imagine a group of n ($n > 0$) girls ranging in integer length from m to $m + n - 1$ cm and a corresponding group of $n + h$ boys ($h \geq 0$) with length ranging from m to $m + n + h - 1$ cm. Clearly m is the minimal length of both boys and girls.

The location is a dancing school. The teacher selects a group of n out of $n + h$ boys. A girl of length l can now choose a dancing partner out of this group of n boys, someone either of her own length or taller up to a maximum of $l + h$.

How many 'matchings' are possible?

The proof of the equivalence of Problem 29 and the Dancing School Problem is left as an excercice.

A Solution

Let's return to the original problem of Lute Kamstra. Let $n > 0$ and $h \geq 0$ and let $A = \{a_1, a_2, ..., a_n\}$ be a subset of $\{1, 2, 3, ..., n + h\}$, with $1 \leq a_1 < a_2 < ... < a_n \leq n + h$. We are looking for permutations π of the elements of A with restrictions on permitted positions such that $k \leq \pi(k) \leq k+h$ for all k. With this restrictions we can associate a (0,1)-matrix $B = [b_{ij}]$, where $b_{ij} = 1$, if and only if a_j is permitted in position i, meaning $i \leq a_j \leq i+h$.

We define S_B as the set of all permitted permutations, to be more precise

$$S_B = \{\pi | \prod_{i=1}^{n} b_{i\pi(i)} = 1\} \tag{32.1}$$

The number of elements of S_B can be calculated by

$$|S_B| = \sum_{\pi} \prod_{i=1}^{n} b_{i\pi(i)} = per(B) \tag{32.2}$$

where $per(B)$ is the permanent of B.

For example, let $n = 4$, $h = 3$ and $A = \{2,3,5,6\}$. We can easily see that in this case we have

$$B = \begin{pmatrix} 1\,1\,0\,0 \\ 1\,1\,1\,0 \\ 0\,1\,1\,1 \\ 0\,0\,1\,1 \end{pmatrix}$$

and $per(B) = 5$, so there are 5 permitted permutations.

Case closed? We know that in general calculating a permanent is a hard problem with algebraic complexity of order $n^2 2^n$. In some special cases there are more efficient algorithms.

Some Questions and Answers

Bipartite Graphs

Matrix B can be interpreted as an incidence matrix of a bipartite graph G with vertices in $X = \{1,2,...,n\}$ and $Y = A = \{a_1, a_2, ..., a_n\}$. An edge of G is a pair (i, a_j) with $b_{ij} = 1$. The edges of the example can be described as

$E = \{(1,2),(1,3),(2,2),(2,3),(2,5),(3,3),(3,5),(3,6),(4,5),(4,6)\}$.
A matching in G is a set of disjoint edges. A perfect matching is a matching containing n edges.

The number of perfect matchings is $per(B)$. Is it possible to calculate the number of perfect matchings with graph theory?

Rook Theory

This is a more algebraic approach of accounting for $|S_B|$. We interpret the matrix B as an $n \times n$ chess board. On squares with $b_{ij} = 1$ we may place a rook. Let $r_k(B)$ be the number of ways we can place k non-attacking rooks on the board (that is, choosing k squares in B no two are on the same line). This corresponds to a bipartite graph G, thus $r_k(B)$ is the number of matchings with k edges.

The rook polynomial $r(B, x)$ is defined as

$$r(B, x) = \sum_{k=0}^{n} r_k(B) x^k$$

So the number of perfect matchings is $r_n(B) = per(B)$.

Is there a simple way to calculate $r(B, x)$ from B? We don't think so, see also the next section.

Configuration Matrix

Let $m = \binom{n+h}{n} = \binom{n+h}{h}$ be the number of different subsets X_i of the set $X = \{1, 2, ..., n+h\}$. We define a (0,1) configuration matrix $C = [c_{ij}]$ with $i = 1, ..., m, j = 1, ..., n+h$ and $c_{ij} = 1$ if and only if $x_j \in X_i$.

The set A in the previous subsection is characterized by the row (0 1 1 0 1 1 0). Is it possible to find a matrix B directly from a row of C?

Let A_k, $k = 1, 2, ..., m$ be a possible subset of X. In the row $[c_{kj}]$ let h be the number of entries with $c_{kj} = 0$, n the number of entries with $c_{kj} = 1$ and $A_k = \{j | c_{kj} = 1\} = \{a_1, a_2, ..., a_n\}$

We define matrix $B_k = [b_{ij}]$ of order n with $b_{ij} = 1$ if and only if $0 \le a_j - i \le h$. So the answer of Problem 29 for A_k is $per(B_k)$ for $k = 1, 2, ..., m$.

Related Poblems

Dancing School and Rooks

What if the girls take over power and put aside the teacher and they choose directly out of the set of $n+h$ boys (accepting the length restrictions)?

Clearly we can once again associate a bipartite graph G to this problem. The n-set X of girls and the $(n+h)$-set Y of boys provide the vertices. If a girl a can choose a boy b of appropriat length we have an edge $\{a, b\}$ of G.

The adjacency matrix A has a special form

$$A = \begin{pmatrix} O & B \\ B^T & O \end{pmatrix}$$

Here B is a (0,1)-matrix of size n by $n+h$ which specifies the adjacencies of the vertices of X and the vertices of Y. We have $b_{ij} = 1$ if and only if $i \leq j \leq i+h$.

A matching M with cardinality n corresponds in the matrix B to a set of n 1's with no two of the 1's on the same line. The total number of matchings with $|M| = n$ is $per(B)$.

It is clear that our problem can be translated into a Rooks Problem:

Find the number of all possible non-attacking placings of n rooks on a $n \times (n+h)$-chessboard, while placing a rook on the i-th row and the j-th column is restricted by the condition $i \leq j \leq i+h$.

Solutions?

Configuration Matrix

We tried to find a recursion from the configuration matrix of the previous section, the so called direct attack. We define the total number of matchings to be $f(n, h)$. We can rearrange the rows of C such that all rows with $c_{i,n+h} = 1$ are placed together. In this case we have $\pi(n) = n + h$, so the corresponding number of matchings is $f(n-1,h)$. In all other rows we have $c_{i,n+h} = 0$, counting for $f(n,h-1)$ matchings, but unfortunately also an extra amount where h comes in! So we can write

$$f(n,h) = f(n-1,h) + f(n,h-1) + x(n,h)$$

So far we are not very succesfull in finding expressions for $x(n,h)$.

In terms of the previous section we may also state

$$f(n,h) = \sum_{k=1}^{m} per(B_k)$$

We think this will not lead to any but trivial solutions, because the calculation of the permanent is a #P-complete problem. The most effective algorithm in general is Ryser's (see later) which is of order of complexity $O(n^2 2^n)$.

Rooks Polynomials

In theory it is possible to calculate the rook polynomial of arbitrary chessboards with the so called expansion theorem. Given a chessboard B, let $r_k(B)$ the number of of ways to put k non-attacking rooks on the board, and let

$$r(B,x) = \sum_{k=0}^{n} r_k(B)x^k$$

be the rook polynomial of board B and $(r_0(B), r_1(B), ..., r_n(B))$ the rook vector of B.

We mark a square on board B as special and denote B_s as the chessboard obtainted from B by deleting the corresponding row and column. B_d is the board obtained from B by deleting the special square. The ways of placing k non-attacking rooks can now be divided in two cases, those that have the rook in the special square and those that have not. In the fist case we have $r_{k-1}(B_s)$ possibilities and in the second $r_k(B_d)$. So we have the relation

$$r_k(B) = r_{k-1}(B_s) + r_k(B_d)$$

This corresponds to

$$r(B, x) = x\, r(B_s, x) + r(B_d, x)$$

This is the so called expansion formula.

Now we can find the rook polynomial of arbitrary boards by applying repeatedly the expansion formula. We think this is only feasible for small sizes, but maybe there are some hidden recursions.

The Permanent to the Rescue

As stated before we do have a solution to our problem: $per(B)$! B has a clear form compared to the previous section. So maybe there are solutions lying around.

There is one in Ryser's Algorithm: Let's try to translate Theorem 7.1.1. of [1][1] to our situation. Let $B = [b_{ij}]$ the $n \times (n+h)$ (0,1)-matrix with $b_{ij} = 1$ if and only if $i \leq j \leq i + h$. Let r be a number with $h \leq r \leq n + h - 1$ and B_r an $n \times (n + h - r)$ sub-matrix of B. We define $\prod(B_r)$ to be the product of the row sums of B_r and $\sum\prod(B_r)$ the sum of all $\prod(B_r)$ taken over all choices of B_r. So

$$per(B) = \sum_{k=0}^{n-1} (-1)^k \binom{h+k}{k} \sum\prod(B_{h+k}) \qquad (32.3)$$

This is a solution, be it not very effective! But maybe we can do better in some cases.

[1] Richard Brualdi and Herbert Ryser. *Combinatorial matrix theory*. Cambridge University Press, Cambridge England New York, 1991. ISBN 0521322650

The complements of ...

Intermezzo: Let A a (0,1)-matrix with m rows and n columns ($m \leq n$). α is a k-subset of the m-set $\{1, 2, ..., m\}$ and β a k-subset of $\{1, 2, ..., n\}$. $A[\alpha, \beta]$ is the $k \times k$ submatrix of A determined by rows i with $i \in \alpha$ and columns j with $j \in \beta$.

The permanent $per(A[\alpha, \beta])$ is called a permanental k-minor of A. We define the sum over all possible α an β

$$p_k(A) = \sum_{\beta}\sum_{\alpha} per(A[\alpha, \beta])$$

We define $p_0(A) = 1$ and note that $p_m(A) = per(A)$. $p_k(A)$ counts for the number of k 1's with no two of the 1's on the same line, so $p_k(A) = r_k(A)$ of the rook vector of A.

According to theorem 7.2.1 of [1] we can evaluate the permanent of a (0,1)-matrix in terms of the permanental minors of the complementary matrix $J_{m,n} - A$, where $J_{m,n}$ is the m by n matrix with all entries 1.

Translated to our matrix B of this section we get

$$per(B) = \sum_{k=0}^{n} (-1)^k p_k(J_{n,n+h} - B) \frac{(n+h-k)!}{h!} \qquad (32.4)$$

This is in particular interesting for $h \geq n-2$, in this case we can easily see that $p_k(J_{n,n+h} - A)$ is independent of h, meaning that $per(B) = f(n,h)$ is polynomial in h. For example we have:

$f(3,h) = h^3 + 3h \ (h \geq 1)$,

$f(4,h) = h^4 - 2h^3 + 9h^2 - 8h + 6 \ (h \geq 2)$,

$f(5,h) = h^5 - 5h^4 + 25h^3 - 55h^2 + 80h - 46 \ (h \geq 3)$,

$f(6,h) = h^6 - 9h^5 + 60h^4 - 225h^3 + 555h^2 - 774h + 484 \ (h \geq 4)$,

$f(7,h) = h^7 - 14h^6 + 126h^5 - 700h^4 + 2625h^3 - 6342h^2 + 9072h - 5840 \ (h \geq 5)$,

We have polynomials up to $f(9,h)$.

Computers are faster now. $n = 12$ or higher maybe feasable.

The Free Dancing School

What if the girls choose directly out of the set of $n+h$ boys and don't accept the length restrictions? They may choose a boy of their own length or taller.

Here again B is a (0,1)-matrix of size n by $n+h$ which specifies the possible dancing pairs. We now have $b_{ij} = 1$ if and only if $i \leq j \leq n+h$. The number of matchings with cardinality n is $per(B)$.

Let $b_1, b_2, ..., b_m$ be integers with $0 \leq b_1 \leq b_2 \leq ... \leq b_m$. The m by b_m (0,1)-matrix $A = [a_{ij}]$ defined by $a_{ij} = 1$ if and only if $1 \leq j \leq b_i$, $(i = 1,2,...,m)$ is called a Ferrers matrix, denoted by

$F(b_1, b_2, ..., b_m)$. According to [1] we can calculate the permanent with

$$per(F(b_1, b_2, ..., b_m)) = \prod_{i=1}^{m}(b_i - i + 1) \qquad (32.5)$$

We can associate B with a Ferrers matrix $F(b_1, b_2, ..., b_n)$ with $b_i = h + i$. So

$$per(B) = \prod_{i=1}^{n}(h + i - i + 1) = (h + 1)^n \qquad (32.6)$$

A result we could also have found by direct counting, but we couldn't resist mentioning Ferrers matrices!

References

[1] R.A Brualdi, H.J. Ryser, Combinational Matrix Theory, Cambridge University Press.

[2] Nieuw Archief voor de Wiskunde (NAW), Problem Section: Problem 29.

Part III

Permanent Questions

33

Note on the Permanent of a Matrix of order n

The Permanent of a Matrix of order n

A new algorithm? Complexity is of the same order as Ryser's algorithm. Multiplication (and addition) with 1 and -1 is extremely easy. So this approach can probably be implemented very efficiently for (0,1) matrices.

Definitions

Let A be a matrix of order n, the permanent of A is defined by

$$per(A) = \sum_{\pi} a_{1\pi(1)} a_{2\pi(2)} ... a_{n\pi(n)} \tag{33.1}$$

while we sum over all $n!$ possible permutations π of $1, 2, ..., n$.

We define a vector $\bar{x} = (x_1, x_2, ..., x_n)^T$ and a vector $\bar{y} = (y_1, y_2, ..., y_n)^T$. Let $\bar{y} = A\bar{x}$. We define a multivariate polynomial

$$\begin{aligned} P(x_1, x_2, ..., x_n) &= \prod_{i=1}^{n} y_i & (33.2) \\ &= (a_{11}x_1 + ... + a_{1n}x_n) \cdot \\ &\quad (a_{21}x_1 + ... + a_{2n}x_n) \cdot \\ &\quad \vdots \\ &\quad (a_{n1}x_1 + ... + a_{nn}x_n) & (33.3) \end{aligned}$$

All terms are of degree n.

In the expansion of (3) we are looking for the coefficient of the term with $x_1 \cdot x_2 \cdot ... \cdot x_n$. When we sum over all possible permutations, we get

$$\sum_{\pi} a_{1\pi(1)} x_{\pi(1)} \cdot a_{2\pi(2)} x_{\pi(2)} \cdot ... \cdot a_{n\pi(n)} x_{\pi(n)} =$$
$$= \left(\sum_{\pi} a_{1\pi(1)} a_{2\pi(2)} ... a_{n\pi(n)} \right) \cdot x_1 x_2 ... x_n$$

So $per(A)$ is the coefficient of the term with $x_1 x_2 ... x_n$.

We define

$$Q(\bar{x}) = (\prod_{i=1}^{n} x_i) \cdot P(x_1, x_2, ..., x_n) \tag{33.4}$$

A Theorem

Now we sum $Q(\bar{x})$ over all possible $\bar{x}$ with $x_i = \pm 1$.

$$\sum_{|\bar{x}|_\infty = 1} Q(\bar{x}) = \sum_{|x_i|=1} (x_1 \cdot x_2 \cdot ... \cdot x_n)\, P(x_1, x_2, ..., x_n)$$

$|\bar{x}|_\infty = 1$ meaning $|x_i| = 1$ for $i = 1, 2, ..., n$.

As we can easily see only the term $per(A) \cdot x_1 x_2 ... x_n$ of $P(x_1, x_2, ..., x_n)$ is always contributing to this sum because $x_i^2 = 1$ for $i = 1, 2, ..., n$. Terms of $Q(x_1, x_2, ..., x_n)$ with factor x_k missing in $P(x_1, x_2, ..., x_n)$, are counted once t and once $-t$ so the overall result is 0. There are 2^n possible vectors $\bar{x}$ with $x_i = \pm 1$, so we have proved:

Theorem 1. *The permanent of A is*

$$per(A) = 2^{-n} \cdot \sum_{|\bar{x}|_\infty = 1} Q(\bar{x}) \tag{33.5}$$

[1] R.A Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, 1991.

Remark: This chapter is essentially the note I sent to the AMM on October 7th, 2003.

34
The Formula of Spies vs Formula of Glynn

Wikipedia

Doing some investigations for this very book, I stumbled on a Wikipedia article Computing the permanent.

https://en.wikipedia.org/, Computing_the_permanent

I found the Balasubramanian–Bax–Franklin–Glynn formula. But wait a minute. Isn't this my very own formula from 2003? Looks almost the same! It is essentially the same!

Figure 34.1: en.wikipedia.org Computing the Permanent

My Formula

In the end of 2002, early 2003 I solved Problem 29 of the NAW (see Chapter 31). The solution was in terms of the permanent of a square (0,1)-matrix. For practical calculations I needed a fast algorithm. An implementation of Ryser's algorithm was one of the possibilities, but I found a formula at least as fast. The derivation was simple, almost elementary. See Chapter 33. I implemented my algorithm in an arbitrary precision C/C++program, which was used to contribute to Neil Sloane's On-line Encyclopedia of Integer Sequences.

For instance https://oeis.org/A087982, https://oeis.org/A088672, https://oeis.org/A089480, https://oeis.org/A089475 and https://oeis.org/A089476

Some of the records still stand! Maybe I should give it a try with the knowledge and computer power of today.

A short history from e-mails

In my e-mail of February 11th, 2003 I showed my results to Robbert Fokkink at that time the editor of the Problem Section of the NAW. In the mean time I had written a short note with proof. How 'new' was the formula? Robbert made the suggestion to send an e-mail to an expert.

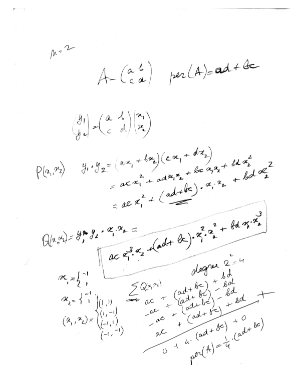
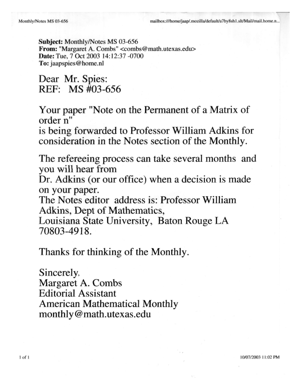


Figure 34.2: Scratch paper

My message to Bruno Codenotti was answered with the suggestion to go straight to the master of permanents: Richard Brualdi. I owned a copy of Brualdi, Ryser, Combinatorial Matrix Theory. This book has a chapter on permanents and was the source of my knowledge of permanents that led to the solution of problem 29. Brualdi and Ryser changed my life! Richard Brualdi wrote: 'Your formula is interesting and I cannot recall seeing it before, but maybe it appears buried in some paper or other.' In a following message he suggested to send the result as a note to the American Mathematical Monthly.

With this recommandation I sent my note to the AMM on October 7th, 2003. In answer came an e-mail with the notification that my note was forwarded to Professor William Adkins for refereeing. The refereeing proces could take several month and I would get an answer from Dr. Adkins from Louisiana State University. No such thing! After more than a year I asked: "What happened to my note? Was it to lightweighted and is it blown with the wind or what?



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Subject: Monthly/Notes MS 03-656
From: "Margaret A. Combs" <combs@math.utexas.edu>
Date: Tue, 7 Oct 2003 14:12:37 -0700
To: jaapspies@home.nl

Dear Mr. Spies:
REF: MS #03-656

Your paper "Note on the Permanent of a Matrix of order n"
is being forwarded to Professor William Adkins for consideration in the Notes section of the Monthly.

The refereeing process can take several months and you will hear from
Dr. Adkins (or our office) when a decision is made on your paper.
The Notes editor address is: Professor William Adkins, Dept of Mathematics,
Louisiana State University, Baton Rouge LA 70803-4918.

Thanks for thinking of the Monthly.

Sincerely,
Margaret A. Combs
Editorial Assistant
American Mathematical Monthly
monthly@math.utexas.edu

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Figure 34.3: e-mail from the AMM

Additionally I mentioned the implementation of my algorithm in the C-language, faster than an optimized Ryser's code. I used this software in some contributions to Neil Sloane's On-Line Encyclopedia of Integer Sequences. See above.

From October 2003 I put my program to work for the calculation of sequences related to permanents. I had some discussions on this theme with Neil Sloane and Edwin Clarke. From Edwin Clarke: "Thanks for the copy of your note on permanents. It is an interesting approach. It might be a good idea to write it up as a new algorithm and compare it to known algorithms.It looks like the time is about 2^n which is certainly better than brute force n!."

In the years that followed I wrote a few things up in an article for the Nieuw Archief voor Wiskunde 5/7 nr. 4 December, 2006: Dancing School problems, Permanent solutions of Problem

29. In a sidenote there is my formula for the permanent of a square matrix. See also the sidenote on The Van der Waerden Conjecture! It lasted almost four years before my solution of Problem 29 was published.

More recent developments

As I told in the introduction to this chapter, I found this formula of Glynn. I was flabbergasted (I like that word!). Wrote an e-mail to my lifeline in Mathematics Robbert Fokkink. Robbert suggested to contact David Glynn in Adelaide. And yes, David replied after reading my long message. I quote: "It looks like you could be added to an increasing list of people that have rediscovered this formula for the permanent. (But the first I think was Balasubramanian in his PhD thesis in India.) My 2013 paper indeed shows that there are very many different formulae that are related via an algebraic structure called the Veronesean. Maybe I can get someone to put a reference to your articles around 2003-2006 or others on the wikipedia website so that it can be noted there."

Once again someone suggested me to write a short note for the Amercan Mathematical Monthly. Again I did. If I may say so, it was a very readable manuscript. With emphasis on the elementary approach. With some history of the permanent. Just in the educational vein. This time the rejection was almost immediate.

The motivation: "In 2010, Glynn published 'The permanent of a square matrix,' a five-page note in The European Journal of Combinatorics that offered four different formulas for the matrix permanent. Now comes this manuscript, which offers an elementary proof of the first of these. The Monthly does publish new, elementary, proofs of old results, but only if the old results are famous, and/or the new proofs are beautiful and insightful. Unfortunately, neither of these applies here; on the contrary, Glynn's proof was more insightful. Consequently, we must decline your submission for publication."

I wrote to David Glynn: "So you win 4-1! And your proof was more 'insightful'. But imagine you have a class of first year students, teaching some graph theory and you come to the permanent. Which method would you prefer giving some insight in the calculation of a permanent?" Do you prefer the insightful: "connected with invariant theory via the polarization identity for a symmetric tensor" or my approach with elementary alghebra?

The first e-mail from David Glynn was also his last one. All my messages ended in a black hole.

SageMath

SAGE, now SageMath , was started in 2005 by William Stein. I discovered SAGE at the end of 2005 and became an early adapter. In 2006 and later I contributed quite some code and more. Sage is Python+, so easy to learn and use. SageMath tries to be an Open Source alternative for the big M's: Magma, Mathematica, Maple and Matlab. And integrates a lot of other Math software.

SageMath: https://www.sagemath.org

My programming skills are a little bit rusty nowadays, but I came up with an implementation of both the formula of Glynn and that of myself in Sage. SageMath has various options to calculate the permanent of matrices over a field. Guess who implemented Ryser's algorithm? In the demoworksheet we calculate the permanent of the example in Glynn's 2010 article. Look for the differences: In Glynn's formule you see row sums in the iterations in mine col sums. This is due to the use of right or left multiplication in matrix theory. Results are the same, of course: the sum of all diagonal products!

Figure 34.4: NAW: A formula for the permanent

And now?

In the NAW nr. 1 March 2020 there is an article "A formula for the permanent". https://www.nieuwarchief.nl/serie5/pdf/naw5-2020-21-1-027.pdf

In the Dutch Wikipedia page on Permanents we now see the Formula of Spies. https://nl.wikipedia.org/wiki/Permanent_(wiskunde)

Figure 34.5: nl.wikipedia.org Permanent (wiskunde)

I still hope there will be an addition to the English version of the Wikipedia page! Volonteers?

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